INTEGRATION OF THE LIFTING FORMULAS AND THE CYCLIC HOMOLOGY OF THE ALGEBRAS OF DIFFERENTIAL OPERATORS

BORIS SHOIKHET

ABSTRACT

We integrate the Lifting cocycles $\Psi_{2n+1}, \Psi_{2n+3}, \Psi_{2n+5}, \dots$ ([Sh1], [Sh2]) on the Lie algebra Dif_n of holomorphic differential operators on an n-dimensional complex vector space to the cocycles on the Lie algebra of holomorphic differential operators on a holomorphic line bundle λ on an n-dimensional complex manifold M in the sense of Gelfand–Fuks cohomology [GF] (more precisely, we integrate the cocycles on the sheaves of the Lie algebras of finite matrices over the corresponding associative algebras). The main result is the following explicit form of the Feigin–Tsygan theorem [FT1]:

$$H^\bullet_{\mathrm{Lie}}(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_n);\mathbb{C}) = \wedge^\bullet(\Psi_{2n+1}, \Psi_{2n+3}, \Psi_{2n+5}, \dots).$$

INTRODUCTION

The cocycles $\Psi_{2n+1}, \Psi_{2n+3}, \Psi_{2n+5}, \dots$ on the Lie algebra $\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_n)$ of finite matrices over (polynomial, holomorphic, formal) differential operators on $\mathbb{C}^n$ ($\Psi_i \in C^i_{\mathrm{Lie}}(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_n);\mathbb{C})$), called *Lifting formulas*, were constructed in the author's works [Sh1], [Sh2].

In the present paper we study the various aspects of the notion of integral in Lie algebra cohomology, applied to the Lie algebra $\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_n)$, namely, we integrate the cocycles $\Psi_{2n+1}, \Psi_{2n+3}, \Psi_{2n+5}, \dots$ on this Lie algebra on an n-dimensional complex manifold M. In this way, we obtain 1-, 3-, 5-, … cocycles on the sheaf of the Lie algebras of holomorphic differential operators in any holomorphic line bundle λ over M (in the sense of Section 3).

0.1. It was proved in [FT1] that the cohomology algebra $H^\bullet(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_n);\mathbb{C})$ is the exterior algebra with the generators in degrees $2n+1, 2n+3, 2n+5, \dots$. This result was proved using the spectral sequence, connecting the Hochschild homology of an associative algebra and its cyclic homology (and the result of [T]); in particular, any explicit formulas for the generators do not follow from this computation.

We prove that

$$H^\bullet(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_n);\mathbb{C}) \simeq \wedge^\bullet(\Psi_{2n+1}, \Psi_{2n+3}, \Psi_{2n+5}, \dots),$$

where $\Psi_{2n+1}, \Psi_{2n+3}, \Psi_{2n+5}, \dots$ are Lifting cocycles (Theorem 4.3.6 in the case $n = 1$ and Theorem 4.4 in the general case.) We prove this theorem using the notion of the *integral* in Lie algebra cohomology, which is due to I.M. Gelfand and D.B. Fuks [GF]. First of all, let us recall the classical construction of the Virasoro 2-cocycle on the Lie algebra $\mathrm{Vect}(S^1)$ of smooth vector fields on the circle.

Date: September 1, 1998.

We choose any (formal) coordinate system in each point $x \in S^1$, smoothly depending on the point x; in this way, we obtain the map $\iota_x \colon \mathrm{Vect}(S^1) \to W_1$, connected with each point $x \in S^1$ (W_1 is the Lie algebra of formal vector field on the line $\mathbb{R}^1$). Let Ψ_3 be the 3-cocycle on the Lie algebra W_1 (in fact, $\dim H^0(W_1;\mathbb{C}) = \dim H^3(W_1;\mathbb{C}) = 1$, $\dim H^i(W_1;\mathbb{C}) = 0$ when $i \neq 0,3$).

We obtain "in any point $x \in S^1$" the cocycle $\Psi_3(x) = \iota_x^*\Psi_3 \in C^3_{\mathrm{Lie}}(\mathrm{Vect}(S^1);\mathbb{C})$. In fact, the cohomological class of all the cocycles $\Psi_3(x)$ is the same, it does not depend on the point $x \in S^1$ and on the choice of the (formal) coordinate system in the point x. It follows from this statement, that there exists a 1-form Θ_2 on S^1 with the values in $C^2_{\mathrm{Lie}}(\mathrm{Vect}(S^1);\mathbb{C})$ such that

$$d_{\mathrm{DR}}\Psi_3(x) = \delta_{\mathrm{Lie}}\Theta_2.$$

Then $\int_{S^1}\Theta_2$ is a cocycle in $C^2_{\mathrm{Lie}}(\mathrm{Vect}(S^1);\mathbb{C})$. Indeed,

$$\delta_{\mathrm{Lie}}\int_{S^1}\Theta_2 = \int_{S^1}\delta_{\mathrm{Lie}}\Theta_2 = \int_{S^1} d_{DR}\Psi_3 = 0.$$

In fact, the choice of Θ_2 is not unique, but there exists in a sense the canonical choice (see Subsec. 2.2). The cohomological class $\left[\int_{S^1}\Theta_2\right]$ does not depend on the choice of the coordinate systems.

Let M be an n-dimensional complex manifold, λ be a holomorphic line bundle on M. Let $\mathrm{Dif}_{\lambda,M}$ be the sheaf of the associative algebras of holomorphic differential operators in λ, and $\mathcal{D}^\bullet_M$ be the Dolbeault complex on M:

$$\mathcal{D}^\bullet_M = \left\{ 0 \to C^\infty_M \xrightarrow{\overline{\partial}} \Omega^{0,1}_M \xrightarrow{\overline{\partial}} \Omega^{0,2}_M \xrightarrow{\overline{\partial}} \dots \right\}.$$

We consider $\Gamma_M(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_{\lambda,M} \otimes_{\mathcal{O}_M} \mathcal{D}^\bullet_M))$ as a DG Lie algebra (see Remark 3.1). When $M = \mathbb{C}^n$, the DG Lie algebra $\Gamma_{\mathbb{C}^n}(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_{\mathbb{C}^n} \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{D}^\bullet_{\mathbb{C}^n}))$ is quasi-isomorphic to $\mathrm{Dif}_n[0]$ as a DG Lie algebra.

We define the cohomology of the sheaf of Lie algebras $\mathrm{Dif}_{\lambda,M}$ (with the bracket $[a,b] = (a*b - b*a)$) as cohomology of the DG Lie algebra $\Gamma_M(\mathrm{Dif}_{\lambda,M}\otimes_{\mathcal{O}_M}\mathcal{D}^\bullet_M)$ and cohomology of the sheaf $\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_{\lambda,M})$ as cohomology of the DG Lie algebra $\Gamma_M(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_{\lambda,M} \otimes_{\mathcal{O}_M} \mathcal{D}^\bullet_M))$. The reason is that complex of sheaves $\mathrm{Dif}_{\lambda,M} \otimes_{\mathcal{O}_M} D^\bullet_M$ is quasi-isomorphic to the sheaf $\mathrm{Dif}_{\lambda,M}[0]$ (as sheaves of associative or Lie algebras).

For any cocycle $\Psi \in C^k_{\mathrm{Lie}}(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_n);\mathbb{C})$ and any singular cycle $\sigma \in C^{\mathrm{sing}}_l(M;\mathbb{C})$ we define (in Sec. 2 and Subsec. 3.2) the *integral* $\int_\sigma \Psi$, which is a *cocycle* in

$$C^{k-l}_{\mathrm{Lie}}(\Gamma_M(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_{\lambda,M}) \otimes_{\mathcal{O}_M} \mathcal{D}^\bullet_M)).$$

The crucial point is that if $[\int_\sigma \Psi] \neq 0$ than $[\Psi] \neq 0$ (here $[\dots]$ stands for the cohomological class) (Theorem 2.3(1)). Therefore the notion of integral gives us an effective tool to prove the cohomological nontriviality of cocycles in $C^\bullet_{\mathrm{Lie}}(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_n);\mathbb{C})$.

Let $n = 1$, $M = \mathbb{C}P^1$. In this simplest case the sheaf of holomorphic differential operators (in any holomorphic line bundle λ) has not higher cohomology (as a sheaf), and the *DG* Lie algebra $\Gamma_{\mathbb{C}P^1}(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_{\lambda,\mathbb{C}P^1}) \otimes_{\mathcal{O}_{\mathbb{C}P^1}} \mathcal{D}^\bullet_{\mathbb{C}P^1})$ is quasi-isomorphic to the Lie algebra of global differential operators $\Gamma(\mathrm{Dif}_{\lambda,\mathbb{C}P^1})[0]$. We will denote the last Lie algebra by $\mathrm{Dif}_{\lambda,\mathbb{C}P^1}$. (The situation is the same for generalized flag varieties, in particular, for projective spaces and flag varieties).

Using the method of [FT1] it is not difficult to show that $H^\bullet_{\text{Lie}}(\mathfrak{gl}^{\text{fin}}_\infty(\text{Dif}_{\mathbb{C}P^1});\mathbb{C})$ is the exterior algebra with the unique generator in dimension 1 and two generators in each dimension $3,5,7,\dots$.

We calculate the integral $\int_{\mathbb{C}P^1}\Psi_{2k+1}\in C^{2k-1}_{\text{Lie}}(\mathfrak{gl}^{\text{fin}}_\infty(\text{Dif}_{\mathbb{C}P^1});\mathbb{C})$ and prove, that it has nonzero value on the $(2k-1)$-cycle in $H_{2k-1}(\mathfrak{gl}^{\text{fin}}_\infty(\text{Dif}_{\mathbb{C}P^1});\mathbb{C})$ which is the image of some $(2k-1)$-cycle in $H_{2k-1}(\mathfrak{gl}^{\text{fin}}_\infty;\mathbb{C})$ under the inclusion $\mathfrak{gl}^{\text{fin}}_\infty\hookrightarrow\mathfrak{gl}^{\text{fin}}_\infty(\text{Dif}_{\mathbb{C}P^1})$ $(A\mapsto A\otimes 1)$ $(k\geq 1)$ (Subsection 4.1–4.3). Then it follows from Theorem 2.3(1) that Ψ_{2k+1} is not cohomologous to zero in $C^\bullet_{\text{Lie}}(\mathfrak{gl}^{\text{fin}}_\infty(\text{Dif}_1;\mathbb{C})$ for any $k\geq 1$. It follows from this result, calculation of [FT1], and the Hopf algebra structure on cohomology $H^\bullet_{\text{Lie}}(\mathfrak{gl}^{\text{fin}}_\infty(\text{Dif}_1);\mathbb{C})$ that

$$H^\bullet_{\text{Lie}}(\mathfrak{gl}^{\text{fin}}_\infty(\text{Dif}_1);\mathbb{C})=\wedge^\bullet(\Psi_3,\Psi_5,\Psi_7,\dots)$$

(in fact, the cocycles Ψ_{2k+1}, $k\geq 1$, are *primitive* elements with respect to the Hopf algebra structure on $H^\bullet_{\text{Lie}}(\mathfrak{gl}^{\text{fin}}_\infty(\text{Dif}_1);\mathbb{C})$).

Unfortunately, we have not found any simple calculation of the integrals using the Dolbeault complex as we described above. Our calculation is based on the Čech resolution, and we define the notion of integral from viewpoint of the Čech resolution (in the case of $\mathbb{C}P^n$) in Subsec. 4.1–4.2. We believe, that both definitions of the integral coincide.

Let $\iota\colon\text{Dif}_{\mathbb{C}P^1}\hookrightarrow\text{Dif}_1$ be the inclusion defined by a choice of coordinate system in a point of $\mathbb{C}P^1$. We prove that

$$H^\bullet_{\text{Lie}}(\mathfrak{gl}^{\text{fin}}_\infty(\text{Dif}_{\mathbb{C}P^1});\mathbb{C})\simeq\wedge^\bullet\left(\int_{\mathbb{C}P^1}\Psi_3;\ \iota^*\Psi_3,\int_{\mathbb{C}P^1}\Psi_5;\ \iota^*\Psi_5,\int_{\mathbb{C}P^1}\Psi_7;\dots\right).$$

Using the integration on $\mathbb{C}P^n$, we prove that

$$H^\bullet_{\text{Lie}}(\mathfrak{gl}^{\text{fin}}_\infty(\text{Dif}_n);\mathbb{C})\simeq\wedge^\bullet(\Psi_{2n+1},\Psi_{2n+3},\Psi_{2n+5},\dots)$$

for any $n\geq 1$. The situation here is quite complicated, because we don't know any way to calculate the integrals directly. Let $\lambda=\mathcal{O}(\boldsymbol{\lambda})$, $\boldsymbol{\lambda}\in\mathbb{Z}$, and we consider

$$\int_{\mathbb{C}P^n}\Psi_{2k+1}\in H^{2k-1}_{\text{Lie}}(\mathfrak{gl}^{\text{fin}}_\infty(\text{Dif}_{\lambda,\mathbb{C}P^n});\mathbb{C}),\qquad k\geq n.$$

For a matrix $(2k-2n+1)$-cycle $\gamma\in H_{2k-2n+1}(\mathfrak{gl}^{\text{fin}}_\infty;\mathbb{C})$ we consider $\int_{\mathbb{C}P^n}\Psi_{2k+1}$ as a polynomial function on $\boldsymbol{\lambda}$. In fact, this is a polynomial of n-th degree. It is not easy to calculate this polynomial, but we calculate its leading coefficient and prove, that it is not equal to zero for some γ. Therefore,

$$\int_{\mathbb{C}P^n}\Psi_{2k+1}\in C^{2k-2n+1}_{\text{Lie}}(\mathfrak{gl}^{\text{fin}}_\infty(\text{Dif}_{\lambda,\mathbb{C}P^n});\mathbb{C})$$

is not cohomologous to zero, and it follows from the Theorem 2.3.(1) that

$$\Psi_{2k+1}\in C^{2k+1}_{\text{Lie}}(\mathfrak{gl}^{\text{fin}}_\infty(\text{Dif}_n);\mathbb{C})$$

is not cohomologous to zero for any $k\geq n$. The remaining part of the proof is the same as in the case $n=1$.

Finally, let us note that the question on the cohomological nontriviality of the pullback $j^*\Psi_{2k+1}$ with respect to the inclusion $j\colon\text{Dif}_n\hookrightarrow\mathfrak{gl}^{\text{fin}}_\infty(\text{Dif}_n)$ $(\mathcal{D}\mapsto E_{11}\otimes\mathcal{D})$ remains open; it solved only in the simplest case $k=n$ ([Sh1], Sect. 2). We don't know how to check the nontriviality of these cocycles because it is not known any explicit formulas for cycles in $C_\bullet(\text{Dif}_n;\mathbb{C})$. So the case of the Lie algebra $\mathfrak{gl}^{\text{fin}}_\infty(\text{Dif}_n)$ turns out more simpler

than the case of the Lie algebra Dif_n itself, because there exist matrix cycles in the case of $\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_n)$.

On the other hand, the values of Lifting formulas itself (not their integrals) on the matrix cycles are equal to 0, as well as the values of the pull-back $\iota^*\Psi_{2k+1}$ on the matrix cycles.

0.2. **Content of the paper.**

Section 1: we recall ([Sh1], [Sh2]) the construction of the Lifting cocycles $\Psi_{2n+1}, \Psi_{2n+3}, \Psi_{2n+5}, \dots$ on the Lie algebra $\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_n)$;

Section 2: we recall the definition of the notion of integral in Lie algebra cohomology in the classical case of the Lie algebra of smooth vector fields on a real manifold; all this material is basically standard ([GF]), the minor modifications are due to Boris Feigin;

Section 3: we give the definition of the integral in the case of the Lie algebra of holomorphic differential operators on a holomorphic line bundle λ on a complex manifold M (more precisely, the sheaf of the Lie algebras, ...); when M is compact, the holomorphic noncommutative residue is defined. In this case, the value of the noncommutative residue on the identity differential operator is equal, up to a constant depending only on M, to the Euler characteristic of the bundle λ (Conjecture 3.3);

Section 4: we calculate the integrals in the case $M = \mathbb{C}P^n$, $n \geq 1$, $\lambda = \mathcal{O}(\boldsymbol{\lambda})$. We use the Čech resolution instead of the Dolbeault resolution of Section 3. We prove here, that $H^\bullet_{\mathrm{Lie}}(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_n); \mathbb{C}) = \wedge^\bullet(\Psi_{2n+1}, \Psi_{2n+3}, \Psi_{2n+5}, \dots)$.

0.3. **Acknowledgements.** I am very much indebted to Boris Feigin, almost all I know about Lie algebra cohomology I learned from the numerous informal discussions with him.

The first ideas of this paper appeared in February 1998, when I worked in the Institut des Hautes Études Scientifiques (Bures–sur–Yvette). I am grateful to the I.H.E.S. for the hospitality and very stimulating atmosphere.

1. Lifting formulas ([Sh1], [Sh2])

1.1. It will be convenient to give the definitions in a bit more generality.

Let $\mathfrak{A}$ be an associative algebra, we will also consider $\mathfrak{A}$ as the Lie algebra with the bracket $[a,b] = a\cdot b - b\cdot a$. Let $\mathrm{Tr}\colon \mathfrak{A} \to \mathbb{C}$ be a trace on the associative algebra $\mathfrak{A}$ (i.e. $\mathrm{Tr}[a,b] = 0$ for any $a, b \in \mathfrak{A}$); we denote by $\mathrm{Der}_{\mathrm{Tr}}\,\mathfrak{A}$ the Lie algebra of all the derivations D of the associative algebra $\mathfrak{A}$ such that $\mathrm{Tr}(Da) = 0$ for any $a \in \mathfrak{A}$.

Lifting formulas are the formulas for $(k+1)$-, $(k+3)$-, $(k+5)$- ... cocycles on the *Lie algebra* $\mathfrak{A}$ constructed by derivations $D_1, \dots, D_k \in \mathrm{Der}_{\mathrm{Tr}}\,\mathfrak{A}$ such that the following conditions (i)–(ii) hold:

(i) $[D_i, D_j] = \mathrm{ad}\, Q_{ij} \quad (Q_{ij} \in \mathfrak{A},\ Q_{ji} = -Q_{ij})$;

(ii) $\underset{i,j,l}{\mathrm{Alt}}\, D_l(Q_{ij}) = 0$

(Let us note that in any case $\underset{i,j,l}{\mathrm{Alt}}\, D_l(Q_{ij})$ lies in the center of the Lie algebra $\mathfrak{A}$.).

In the simplest case, when $Q_{ij} = 0$ for all i, j the formula for $(k+1)$-cocycle is given by the following formula:

$$\Psi_{k+1}(A_1, \dots, A_{k+1}) = \operatorname*{Alt}_{A_1,\dots,A_{k+1}} \operatorname*{Alt}_{D_1,\dots,D_k} \operatorname{Tr}(D_1A_1 \cdot \ldots \cdot D_kA_k \cdot A_{k+1}). \tag{1}$$

In the general case, the r.h.s. of (1) is the "leading term" of the $(k+1)$-cocycle, also exist terms linear, quadratic, ..., $\left[\frac{k}{2}\right]$-th degree in Q_{ij}.

Let $\mathfrak{A} = \Psi\mathrm{Dif}_n$ be the associative algebra of the formal pseudo-differential operators on $(S^1)^n$ ([A]). One can define $2n$ (exterior) derivations of this algebra: $\operatorname{ad}(\ln x_1)$, ..., $\operatorname{ad}(\ln x_n)$; $\operatorname{ad}(\partial_1)$, ..., $\operatorname{ad}(\ln \partial_n)$. We use the formal symmetry between the symbols ∂_i and x_i arising from the generating relation $[\partial_i, x_i] = 1$ to define $\operatorname{ad}(\ln \partial_i)$. These derivations as well as the corresponding 2-cocycles firstly appear in [KK]. The trace on $\Psi\mathrm{Dif}_n$ is the "noncommutative residue" ([A]): it is defined as the coefficient at the term $x_1^{-1} \dots x_n^{-1}\partial_1^{-1} \dots \partial_n^{-1}$ in any coordinate system; it was proved in [A], that $\operatorname{Tr}[a,b] = 0$ for any $a, b \in \mathfrak{A}$. It is easy to see ([Sh1]) that in fact $\operatorname{ad}(\ln x_i), \operatorname{ad}(\ln \partial_i) \in \operatorname{Der}_{\operatorname{Tr}} \Psi\mathrm{Dif}_n$. It was proved in [Sh1] that

$$[\operatorname{ad}(\ln \partial_i), \operatorname{ad}(\ln \partial_i)] = \operatorname{ad}\left(x_i^{-1}\partial_i^{-1} + \frac{1}{2}x_i^{-2}\partial_i^{-2} + \frac{2}{3}x_i^{-3}\partial_i^{-3} + \ldots + \frac{(n-1)!}{n}x_i^{-n}\partial_i^{-n} + \ldots\right). \tag{2}$$

Therefore, the set of $2n$ derivations

$$\{\operatorname{ad}(\ln x_1), \dots, \operatorname{ad}(\ln x_n); \operatorname{ad}(\ln \partial_1), \dots, \operatorname{ad}(\ln \partial_n)\}$$

satisfy the conditions (i), (ii) above.

We can reply this construction in the case $\mathfrak{A} = \mathfrak{gl}_\infty^{\mathrm{fin}}(\Psi\mathrm{Dif}_n)$, here

$$\operatorname{Tr}_{\mathfrak{gl}_\infty^{\mathrm{fin}}(\Psi\mathrm{Dif}_n)} = \operatorname{Tr}_{\Psi\mathrm{Dif}_n} \circ \operatorname{Tr}_{\mathfrak{gl}_\infty^{\mathrm{fin}}}.$$

The derivations $\operatorname{ad}(\ln x_1)$, ..., $\operatorname{ad}(\ln x_n)$; $\operatorname{ad}(\ln \partial_1)$, ..., $\operatorname{ad}(\ln \partial_n)$ act in the obvious way on the algebra $\mathfrak{gl}_\infty^{\mathrm{fin}}(\Psi\mathrm{Dif}_n)$, and the conditions (i)–(ii) hold again.

Remark. Strictly speaking, $[\operatorname{ad}(\ln \partial_i), \operatorname{ad}(\ln x_i)] = \operatorname{ad} Q$, where Q is an infinite matrix in the case of the Lie algebra $\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_n)$.

By definition, the Lifting formulas for the algebra Dif_n are pull-backs of the Lifting formulas for $\Psi\mathrm{Dif}_n$ with respect to the natural imbedding $\mathrm{Dif}_n \hookrightarrow \Psi\mathrm{Dif}_n$, and also for the Lifting formulas in the case of the algebra $\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_n)$.

We construct the Lifting formulas for $\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_1)$ $(k = 2)$ in Subsection 1.2 and for general k in Subsection 1.3.

1.2. **The case $n = 1$ ([Sh1], [KLR]).** Let $k = 2$, so we have two derivations D_1, D_2 such that $[D_1, D_2] = \operatorname{ad} Q$.

The simplest Lifting formula for 3-cocycle is given by the following formula:

$$\Psi_3(A_1, A_2, A_3) = \operatorname*{Alt}_{A,D} \operatorname{Tr}(D_1A_1 \cdot D_2A_2 \cdot A_3) + \operatorname*{Alt}_{A} \operatorname{Tr}(Q \cdot A_1 \cdot A_2 \cdot A_3) \tag{3}$$

In the general case, for $i = 2, 3, 4, \dots$

$$(4)\quad \Psi_{2i+1}(A_1, \dots, A_{2i+1}) = \operatorname*{Alt}_{A} \operatorname{Tr}(Q \cdot A_1 \cdot A_2 \cdot \ldots \cdot A_{2i+1}) +$$
$$+ \operatorname*{Alt}_{A,D} \operatorname{Tr}(D_1 A_1 \cdot D_2 A_2 \cdot A_3 \cdot \ldots \cdot A_{2i+1} + D_1 A_1 \cdot A_2 \cdot A_3 \cdot D_2 A_4 \cdot A_5 \cdot \ldots \cdot A_{2i+1} +$$
$$+ D_1 A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5 \cdot D_2 A_6 \cdot A_7 \cdot \ldots \cdot A_{2i+1} + \ldots +$$
$$+ a_s \cdot D_1 A_1 \cdot A_2 \cdot \ldots \cdot A_s \cdot D_2 A_{s+1} \cdot A_{s+2} \cdot \ldots \cdot A_{2i+1})$$
$$\text{where} \qquad \begin{cases} s = i+1, \quad a_s = \dfrac{1}{2} & \text{if } i \text{ is even} \\ s = i, \quad a_s = 1 & \text{if } i \text{ is odd} \end{cases}.$$

Theorem ([Sh1], Sect. 1, [KLR])**.** *The formulas for $\Psi_3, \Psi_5, \Psi_7, \dots$ given by* (3), (4) *define cocycles on the Lie algebra $\mathfrak{A}$.* □

1.3. **The general case.** In this Subsection we suppose that $D_1, \dots, D_k \in \operatorname{Der}_{\operatorname{Tr}} \mathfrak{A}$ satisfy conditions (i), (ii) from Subsec. 1.1.

First of all, let us write the formula for 5-cocycle in the case $k = 4$ (this is the simplest Lifting formula in this case):

$$(5)\quad \Psi_5(A_1, A_2, A_3, A_4, A_5) =$$
$$\begin{aligned} = \operatorname*{Alt}_{A,D} \operatorname{Tr}\{ & D_1 A_1 \cdot D_2 A_2 \cdot D_3 A_3 \cdot D_4 A_4 \cdot A_5 \\ & \left.\begin{aligned} &+ A_1 \cdot Q_{12} \cdot A_2 \cdot D_3 A_3 \cdot D_4 A_4 \cdot A_5 \\ &+ D_1 A_1 \cdot A_2 \cdot Q_{23} \cdot A_3 \cdot D_4 A_4 \cdot A_5 \\ &+ D_1 A_1 \cdot D_2 A_2 \cdot A_3 \cdot Q_{34} \cdot A_4 \cdot A_5 \end{aligned}\right] \quad \text{terms, linear in } Q_{ij} \\ & \left.+ A_1 \cdot Q_{12} \cdot A_2 \cdot A_3 \cdot Q_{34} \cdot A_4 \cdot A_5\} \right] \quad \text{term, quadratic in } Q_{ij} \end{aligned}$$

We don't alternate the symbols i, j in Q_{ij}.

The next step is the formula for $(k+1)$-cocycle for any k. We need some preparations. We consider the interval of the length $k-2$ with some *marked* points, such that the *distance between any two marked points is greater or equal than* 2. Let us denote by I_l the set of all such marked intervals with l marked points ($1 \le l \le \left[\frac{k}{2}\right]$). Denote by $1, \dots, k-1$ the integral points of the interval.

Definition. Suppose that $t \in I_l$ and $i_1 < \ldots < i_l$ are its marked points ($1 \le i_1,\ i_k \le \left[\frac{k}{2}\right]$ and $i_{s+1} - i_s \ge 2$ for all $s = 1, \dots, l-1$). Then

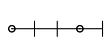
$$\mathcal{O}(t) = \operatorname*{Alt}_{A,D} \operatorname{Tr}(P_{1,t} \cdot \ldots \cdot P_{k+1,t})$$

where

$$\left.\begin{aligned} P_{j,t} &= A_j \cdot Q_{j,j+1} \\ P_{j+1,t} &= A_{j+1} \end{aligned}\right\} \qquad \text{when } j = i_s \text{ for some } s = 1, \dots, l, \text{ i. e., if point } j \text{ is marked}$$

$$P_{j,t} = D_j A_j \qquad \text{when } j \text{ and } j-1 \text{ are not marked and } j \ne k+1$$

$$P_{k+1,t} = A_{k+1}$$

Example. If $k = 6$ and $t =$ , $t \in I_2$ then

$$\mathcal{O}(t) = \operatorname*{Alt}_{A,D} \operatorname{Tr}(A_1 \cdot Q_{12} \cdot A_2 \cdot D_3 A_3 \cdot D_4 A_4 \cdot D_5 A_5 \cdot D_6 A_6 \cdot A_7).$$

If $t = \circ\!\!-\!\!+\!\!-\!\!\circ\!\!-\!\!+\!\!-\!\!\circ$, $t \in I_3$, then

$$\mathcal{O}(t) = \operatorname*{Alt}_{A,D} \operatorname{Tr}(A_1 \cdot Q_{12} \cdot A_2 \cdot A_3 \cdot Q_{34} \cdot A_4 \cdot A_5 \cdot Q_{56} \cdot A_6 \cdot A_7).$$

We don't alternate the symbols i, j in Q_{ij} in this definition.

Theorem. *Let $\Sigma_l = \sum\limits_{t \in I_l} \mathcal{O}(t)$. Then*

$$\Psi_{k+1} = \operatorname*{Alt}_{A,D} \operatorname{Tr}(D_1 A_1 \cdot D_2 A_2 \cdot \ldots \cdot D_k A_k \cdot A_{k+1}) + \Sigma_1 + \Sigma_2 + \ldots + \Sigma_{\left[\frac{k}{2}\right]} \tag{6}$$

is a $(k+1)$-cocycle on the Lie algebra $\mathfrak{A}$.

□

This Theorem was proven in [Sh2], Section 3.

It remains to define the cocycles $\Psi_{k+3}, \Psi_{k+5}, \Psi_{k+7}, \ldots$ (for k derivations $D_1, \ldots, D_k$). First of all, let us suppose that $Q_{ij} = 0$ for all i, j.

Definition. The set a^s_{even} is the set of the sequences $\{a_i\}$ of the length $k + 2s$ such that:

(i) $a_i \in \{0, 1\}$ for all $i = 1, \ldots, k + 2s$;

(ii) $a_1 = 1$;

(iii) k of the a_i's are equal to 1 and $2s$ of the a_i's are equal to 0.

(iv) the number of 0's between the two nearest 1's is *even*; this condition also should hold for the "tail" of the sequence $\{a_i\}$, as if the number's a_i were placed on a circle.

We define the $(k + 2s - 1)$-cochain $R_{a_1,\ldots,a_{k+2s}}(A_1, \ldots, A_{k+2s-1})$ for each $\{a_i\} \in a_{\text{even}}$. Roughly speaking, we just shorten any sequence of consecutive zeros to a single zero; after this procedure the sequence will have odd length. We choose the first such sequence of the consecutive 0's. More precisely, let us define the sequence $\{\widetilde{a}_i\} \in a_{\text{even}}$ in the following way:

let $s_1 = \min\limits_i (a_i = 0)$, $s_2 = \max\limits_{i > s_1} (a_i = 1)$; then:

$$\begin{aligned}
&\widetilde{a}_1, \ldots, \widetilde{a}_{s_1 - 1} = 1;\\
&\widetilde{a}_{s_1}, \ldots, \widetilde{a}_{s_2 - 2} = 0;\\
&\widetilde{a}_i = a_{i+1} \quad \text{for} \quad s_2 - 1 \le i \le k + 2s - 1.
\end{aligned}$$

Definition.

$$R_{a_1,\ldots,a_{k+2s}}(A_1, \ldots, A_{k+2s-1}) = \operatorname*{Alt}_{A,D} \operatorname{Tr}(P_1 \cdot \ldots \cdot P_{k+2s-1}),$$

where

$$\begin{aligned}
&P_i = D_{j(i)} A_i \quad \text{for} \quad \widetilde{a}_i = 1,\\
&P_i = A_i \quad \text{for} \quad \widetilde{a}_i = 0
\end{aligned}$$

and $j(i)$ is defined in the following way: $j(1) = 1$ and $j(i_1) < j(i_2)$ when $i_1 < i_2$ and $j = 1, \ldots, n$. In other words, j takes values from 1 to k *in turn*.

Theorem ([Sh2], Section 1)**.**

$$\Psi^0_{k+2s-1} = \sum_{\{a_i\} \in a_{\text{even}}} (-1)^{s_1} R_{a_1,\ldots,a_{k+2s}}(A_1, \ldots, A_{k+2s-1}) \tag{7}$$

is a $(k + 2s - 1)$-cocycle on the Lie algebra $\mathfrak{A}$ when all $Q_{ij} = 0$.

□

The formula (7) is the generalization of the formula (1) on the case of the higher cocycles.

Let us consider the general case, $Q_{ij} \neq 0$. We quantize each summand $R_{a_1,\dots,a_{k+2s}}$ separately.

Definition. (i) Denote by $\mathrm{Circle}_l^{a_1,\dots,a_{k+2s}}$ the set of all circles with $k+2s-1$ integral points from which l $(1 \le l \le \left[\frac{k}{2}\right])$ are marked. The distance between any two marked points is ≥ 2. The points are enumerated by $1,\dots,k+2s-1$. Point i may be marked only if $\widetilde{a}_i = 1$ and $\widetilde{a}_{i+1} = 1$ (we suppose that $\widetilde{a}_{k+2s} = \widetilde{a}_1$).

(ii) for $t \in \mathrm{Circle}_l^{a_1,\dots,a_{k+2s}}$ we define $\mathcal{O}(t)$ by the analogy with the definition above after replacing the interval with the circle. We don't alternate the symbols i, j in Q_{ij} in this definition.

(iii)

$$\Sigma_l^{a_1,\dots,a_{k+2s}} = \sum_{t \in \mathrm{Circle}_l^{a_1,\dots,a_{k+2s}}} \mathcal{O}(t)$$

Theorem ([Sh2], Section 3)**.** *The formula*

$$\Psi_{k+2s-1} = \Psi^0_{k+2s-1} + \sum_{\{a_i\} \in a_{\mathrm{even}}} (-1)^{s_1} \sum_{l \ge 1} \Sigma_l^{a_1,\dots,a_{k+2s}} \tag{8}$$

defines a $(k+2s-1)$-cocycle on the Lie algebra $\mathfrak{A}$, where Ψ^0_{k+2s-1} is defined by the formula (7).

□

2. Integration in the Lie algebra cohomology [GF]

2.1. **C^∞-case.** Let M be an n-dimensional C^∞-manifold, let W_n be the Lie algebra of C^∞-vector fields on $\mathbb{R}^n$ and let $\mathrm{Vect}(M)$ be the Lie algebra of C^∞-vector fields on M.

The integration is a procedure, corresponding to any cocycle $\xi \in C^k_{\mathrm{Lie}}(W_n;\mathbb{C})$ and any singular cycle $\sigma \in C_l^{\mathrm{sing}}(M,\mathbb{C})$ the cocycle $\int_\sigma \xi \in C^{k-l}_{\mathrm{Lie}}(\mathrm{Vect}(M);\mathbb{C})$ such that:

(i) $\left[\int_\sigma \xi\right] = 0 \quad \text{if } [\xi] = 0;$

(ii) $\left[\int_\sigma \xi\right] = 0 \quad \text{if } [\sigma] = 0$

(here $[\dots]$ denotes the cohomological class).

Let $x \in M$ be any point of M, and let $\varphi\colon U \to M$ be any coordinate system in the point x (U is a neighbourhood of 0 in $\mathbb{R}^n$ and $\varphi(0) = x$). The map φ induces the map $\varphi_{\mathrm{Vect}}\colon \mathrm{Vect}(M) \to W_n$, and the corresponding map of the cochain complexes:

$$\varphi^*_{\mathrm{Vect}}\colon C^\bullet_{\mathrm{Lie}}(W_n;\mathbb{C}) \to C^\bullet_{\mathrm{Lie}}(\mathrm{Vect}(M);\mathbb{C}).$$

Lemma. *Let $\xi \in C^\bullet_{\mathrm{Lie}}(W_n;\mathbb{C})$ be a cocycle. Then the cohomological class $[\varphi^*_{\mathrm{Vect}}(\xi)]$ does not depend on the choice of the point $x \in M$ and of the coordinate system $\varphi\colon U \to M$.*

Proof. The ad-action of the Lie algebra $\mathrm{Vect}(M)$ moves points of M and induces the trivial action in cohomology. □

Let $\xi \in C^k_{\mathrm{Lie}}(W_n;\mathbb{C})$ be a cocycle. We choose a coordinate system $\varphi_x\colon U \to M$ ($\varphi_x(0) = x$) in all the points $x \in M$, *smoothly depending on the point* x; by definition,

$$\xi(x) = \varphi^*_{x,\mathrm{Vect}}(\xi) \in C^k_{\mathrm{Lie}}(\mathrm{Vect}(M);\mathbb{C}).$$

According to Lemma, all the cocycles $\xi(x)$ ($x \in M$) are cohomologous to each other. Therefore, there exists an element $\Theta_1 \in \Omega^1_M \otimes C^{k-1}_{\mathrm{Lie}}(\mathrm{Vect}(M);\mathbb{C})$ such that $d_{\mathrm{DR}}\xi(x) = \delta_{\mathrm{Lie}}\Theta_1$ (here d_{DR} denotes the de Rham differential and δ_{Lie} denotes the differential in the cochain complex).

Furthermore, we will find elements $\Theta_i \in \Omega^i_M \otimes C^{k-i}_{\mathrm{Lie}}(\mathrm{Vect}(M);\mathbb{C})$ such that

$$\begin{cases} d_{\mathrm{DR}}\xi(x) = \delta_{\mathrm{Lie}}\Theta_1 \\ d_{\mathrm{DR}}\Theta_1 = \delta_{\mathrm{Lie}}\Theta_2 \\ \dots\dots\dots\dots\dots\dots \\ d_{\mathrm{DR}}\Theta_{n-1} = \delta_{\mathrm{Lie}}\Theta_n \end{cases}. \tag{9}$$

It is obvious that if solution of (9) exist, it is not unique.

Proposition. *For any singular cycle $\sigma \in C^{\mathrm{sing}}_l(M;\mathbb{C})$ and for any solution $\{\Theta_i\}$ of the system* (9) *the integral of l-form*

$$\int_\sigma \Theta_l \in C^{k-l}_{\mathrm{Lie}}(\mathrm{Vect}(M);\mathbb{C})$$

is a cocycle.

Proof.

$$\delta_{\mathrm{Lie}}\int_\sigma \Theta_l = \int_\sigma \delta_{\mathrm{Lie}}\Theta_l = \int_\sigma d_{\mathrm{DR}}\Theta_{l-1} \overset{\text{by Stokes formula}}{=} 0. \quad \square$$

It is obvious (see Subsec. 2.3) that the condition (ii) holds for any solution of the system (9). We have in mind (but we don't make the exact statement) that solution of (9) is unique if we require the execution of the condition (i).

2.2. **The solution of (9).** In this Subsection we construct the canonical solution of the system (9).

Let $x \in M$ and we have choose a coordinate system in any point, smoothly depending on the point. Then any tangent vector $v \in T_xM$ in the point x determines an infinitesimal 1-parametric group of diffeomorphisms on $\mathbb{R}^n$, i.e. the element $t_v \in W_n$.

In the cochain complex of any Lie algebra $\mathfrak{g}$ the following identity holds:

$$\delta_{\mathrm{Lie}} \circ \iota_t \pm \iota_t \circ \delta_{\mathrm{Lie}} = \mathrm{ad}(t) \tag{10}$$

($t \in \mathfrak{g}$ and $\iota_t \colon C^{\bullet}_{\mathrm{Lie}}(\mathfrak{g};\mathbb{C}) \to C^{\bullet-1}_{\mathrm{Lie}}(\mathfrak{g};\mathbb{C})$ is the substitution of t for the first argument).

Let $\widetilde{t_v}$ be any C^∞-vector field on M such that $\widetilde{t_v}|_{\varphi}(u) = \varphi_{x,\mathrm{Vect}}(t_v)$. It follows directly from (10) that

$$\delta_{\mathrm{Lie}} \circ \iota_{\widetilde{t_v}} \pm \iota_{\widetilde{t_v}} \circ \delta_{\mathrm{Lie}} = \mathrm{ad}(\widetilde{t_v}) \tag{11}$$

in the cochain complex $C^{\bullet}_{\mathrm{Lie}}(\mathrm{Vect}(M);\mathbb{C})$.

Lemma.

$$(d_{\mathrm{DR}}\xi(x))v = \mathrm{ad}\,\widetilde{t_v}(\xi(x)). \tag{12}$$

Remark. The r.h.s. of (12) does not depend on prolongation $\widetilde{t_v}$ of vector field t_v.

Proof. The derivation in the direction of the vector v corresponds to the infinitesimal 1-parametric group of diffeomorphisms and $\mathrm{ad}(\widetilde{t_v})$ corresponds to the Ad-action of the diffeomorphism. $\square$

It follows from (11), (12) that

$$(d_{\mathrm{DR}}\xi(x))v = \delta_{\mathrm{Lie}} \circ \iota_{\widetilde{t}_v}\xi(x)$$

or, in other words, if we set

$$\Theta_1(v)(A_1, \dots, A_{k-1}) = \xi(x)(\widetilde{t}_v, A_1, \dots, A_{k-1}) \tag{13}$$

then $(d_{\mathrm{DR}}\xi(x))v = \delta_{\mathrm{Lie}}\Theta_1(v)$.

It is obvious that Θ_1 does not depend on the prolongation $\widetilde{t}_v$ of the vector field t_v.

Furthermore, let us set

$$\Theta_i(v_1, \dots, v_i)(A_1, \dots, A_{k-i}) = \xi(x)(\widetilde{t}_{v_1}, \dots, \widetilde{t}_{v_i}, A_1, \dots, A_{k-i}). \tag{14}$$

Theorem. *The elements* $\{\Theta_i\}$ *defined by the formula* (14), $\Theta_i \in \Omega^i_M \otimes C^{k-i}_{\mathrm{Lie}}(\mathrm{Vect}(M);\mathbb{C})$, *gives us a solution of the system* (9).

Proof. We have proved the statement in the case $i = 1$. We prove in the case $i = 2$, the general case in analogous.

By the Cartan formula, we have:

$$d_{\mathrm{DR}}(\Theta_1)(v_1(x), v_2(x)) = \Theta_1([v_1, v_2](x)) - v_1(\Theta_1(v_2(x)) + v_2(\Theta_1(v_1(x))). \tag{15}$$

In the formula (15) v_1 and v_2 are vector fields such that $v_i|_x = v_i(x)$.

According to Lemma, we have:

$$v_1(\Theta_1(v_2))(x) = \mathrm{ad}(\widetilde{t}_{v_1})(\Theta_1(v_2(x)), \tag{16}$$

$$v_2(\Theta_1(v_1))(x) = \mathrm{ad}(\widetilde{t}_{v_2})(\Theta_1(v_1(x)) \tag{17}$$

if we choose the vector fields v_1 and v_2 such that $[v_1, v_2] = 0$. Note also that $[\widetilde{t}_{v_1(x)}, \widetilde{t}_{v_2(x)}] = 0$ for any two vectors $v_1(x), v_2(x)$. Therefore

$$d_{\mathrm{DR}}(\Theta_1)(v_1, v_2) = -[\widetilde{t}_{v_1}, \xi(x)(\widetilde{t}_{v_2}, A_1, \dots, A_{k-1})] + [\widetilde{t}_{v_2}, \xi(x)(t_{v_1}, A_1, \dots, A_{k-1})]. \tag{18}$$

Now it follows from the cocycle condition for $\xi(x)$ that the r.h.s. of (18) is equal to $\delta_{\mathrm{Lie}}\Theta_2(v_1, v_2)$. ∎

2.3. Conditions (i) and (ii).

Theorem. (1) *If* $[\xi] = 0$ *then* $\left[\int_\sigma \xi\right] = 0$ *for any manifold* M *and* $\sigma \in C^{\mathrm{sing}}_\bullet(M;\mathbb{C})$.

(2) *If* $[\sigma] = 0$ *then* $\left[\int_\sigma \xi\right] = 0$ *for any* ξ.

Proof. (1): Let $\xi = \delta_{\mathrm{Lie}}\eta$. Then

$$\Theta_i(v_1, \dots, v_i)(A_1, \dots, A_{k-i}) = (\delta_{\mathrm{Lie}}\eta)(\widetilde{t}_{v_1}, \dots, \widetilde{t}_{v_i}, A_1, \dots, A_{k-i}).$$

We have:

$$(\delta_{\mathrm{Lie}}\eta)(\widetilde{t}_{v_1}, \dots, \widetilde{t}_{v_i}, A_1, \dots, A_{k-i}) = \delta_{\mathrm{Lie}}(\eta(\widetilde{t}_{v_1}, \dots, \widetilde{t}_{v_i}, A_1, \dots, A_{k-i})$$

modulo the following expressions:

(i) $\mathrm{ad}(\widetilde{t}_{v_j})\eta(\widetilde{t}_{v_1}, \dots, \widehat{\widetilde{t}_{v_j}}, \dots, \widetilde{t}_{v_i}, A_1, \dots, A_{k-i})$;

(ii) $\eta([\widetilde{t}_{v_\alpha}, \widetilde{t}_{v_\beta}], \widetilde{t}_{v_1}, \dots, \widehat{\widetilde{t}_{v_\alpha}}, \dots, \widehat{\widetilde{t}_{v_\beta}}, \dots, \widetilde{t}_{v_i}, A_1, \dots, A_{k-i})$.

By formula (12), the summands of the type (i) are equal to $(d_{\mathrm{DR}}\eta)(v_j)$ and therefore their pull-backs on the singular cycle σ are equal to 0.

On the other hand, $[\widetilde{t}_{v_\alpha}, \widetilde{t}_{v_\beta}] = 0$ for any v_α and v_β, because the corresponding 1-parametric groups commute with each other.

Therefore, the summands of both types (i) and (ii) are equal to 0 after pull-backs on σ, and

$$\int_\sigma \Theta_i = \int_\sigma \delta_{\mathrm{Lie}}(\eta(\widetilde{t}_{v_1},\dots,\widetilde{t}_{v_i},A_1,\dots,A_{k-i})) = \delta_{\mathrm{Lie}}\int_\sigma \eta(\widetilde{t}_{v_1},\dots,\widetilde{t}_{v_i},A_1,\dots,A_{k-i}).$$

(2): $\int_{\partial\gamma}\Theta_i = \int_\gamma d_{\mathrm{DR}}\Theta_i = \int_\gamma \delta_{\mathrm{Lie}}\Theta_{i+1} = \delta_{\mathrm{Lie}}\int_\gamma \Theta_{i+1}$. □

Corollary. *The cohomological class of the integral $\int_\sigma \Theta_i$ does not depend on the choice of the coordinate systems (σ is a singular i-cycle in M).*

Proof. It follows directly from Theorem 2.3.(1) and Lemma 2.1. □

3. Integration in the Holomorphic Case

3.1. Extended Lifting Formulas.

3.1.1. Let us denote by $\mathcal{D}_M^\bullet$ the Dolbeault complex of the structure sheaf $\mathcal{O}_M$ of a complex manifold M,

$$\mathcal{D}_M^\bullet = \left\{ 0 \to C_M^\infty \xrightarrow{\overline{\partial}} \Omega_M^{0,1} \xrightarrow{\overline{\partial}} \Omega_M^{0,2} \xrightarrow{\overline{\partial}} \dots \right\};$$

it is clear, that $\mathcal{D}_M^\bullet$ is a complex of sheaves of $\mathcal{O}$-modules. We consider the complex of global sections $\Gamma_M(\mathcal{D}_M^\bullet)$ as a super-commutative associative DG algebra.

Let us denote by $\mathrm{Dif}_{\lambda,M}$ the sheaf of holomorphic differential operators in a holomorphic line bundle λ on M, then the "Dolbeault complex of the sheaf $\mathrm{Dif}_{\lambda,M}$" is defined as $\mathrm{Dif}_{\lambda,M}\otimes_{\mathcal{O}_M}\mathcal{D}_M^\bullet$. Also the Dolbeault complex $\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\lambda,M})\otimes_{\mathcal{O}_M}\mathcal{D}_M^\bullet$ is defined. On the complex of global sections $\Gamma_M(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\lambda,M})\otimes_{\mathcal{O}_M}\mathcal{D}_M^\bullet)$ the structure of DG Lie algebra is defined. In the case $M=\mathbb{C}^n$ or $M=\mathbb{C}P^n$ the DG Lie algebra $\Gamma_M(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\lambda,M})\otimes_{\mathcal{O}_M}\mathcal{D}_M^\bullet)$ is quasi-isomorphic to the (DG) Lie algebra $\mathfrak{gl}_\infty^{\mathrm{fin}}(\Gamma_M(\mathrm{Dif}_{\lambda,M}))[0]$ as a DG Lie algebra.

Remark. Let $\mathfrak{g}$ be a Lie algebra, and let $m^\bullet$ be super-commutative associative DG algebra. Then on $\mathfrak{g}\otimes_{\mathbb{C}} m^\bullet$ the structure of DG Lie algebra is defined:

$$[g_1\otimes m_1, g_2\otimes m_2] = [g_1,g_2]\otimes m_1m_2.$$

If $\mathfrak{g}$ is the Lie algebra constructed from an associative algebra $\mathfrak{g}$ (with the bracket $[a,b]=a*b-b*a$) then except the above construction, $\mathfrak{g}\otimes m^\bullet$ is an associative DG algebra with the product $(g_1\otimes m_1)*(g_2\otimes m_2)=(g_1*g_2)\otimes(m_1m_2)$, and there arises the structure of purely even Lie algebra on

$$g\otimes m^\bullet\colon [g_1\otimes m_1, g_2\otimes m_2] = (g_1*g_2 - (-1)^{\deg m_1\cdot \deg m_2} g_2*g_1)\otimes(m_1m_2).$$

Everywhere in this work we have in mind the *first* construction, and we consider $\mathfrak{g}\otimes m^\bullet$ as DG *Lie algebra*.

3.1.2. In the case $M=\mathbb{C}^n$ the DG Lie algebra $\Gamma_{\mathbb{C}^n}(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\mathbb{C}^n})\otimes_{\mathcal{O}_{\mathbb{C}^n}}\mathcal{D}_{\mathbb{C}^n}^\bullet)$ is quasi-isomorphic to the DG Lie algebra $\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_n)[0]$, and there exists the canonical imbedding of DG Lie algebras:

$$\iota\colon \mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_n)[0] \hookrightarrow \mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_n)\otimes_{\mathcal{O}_{\mathbb{C}^n}}\mathcal{D}_{\mathbb{C}^n}^\bullet.$$

We want to find formulas for the "extended" Lifting cocycles $\widehat{\Psi}_{2k+1}$ ($k \geq n$) of the DG Lie algebra $\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_n) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{D}^\bullet_{\mathbb{C}^n}$ such that $\iota^*(\widehat{\Psi}_{2k+1}) = \Psi_{2k+1}$. To do this, let us note that there exists another map of DG Lie algebras:

$$p\colon \mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_n) \otimes_{\mathcal{O}_{\mathbb{C}^n}} \mathcal{D}^\bullet_{\mathbb{C}^n} \to \mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_n)[0].$$

By the definition, $p = 0$ in the degrees $\neq 0$, and

(19) $p(\mathcal{D} \otimes_{\mathcal{O}} f(z_1, \dots, z_n, \overline{z}_1, \dots, \overline{z}_n) = D \otimes_{\mathcal{O}} f(z_1, \dots, z_n, 0, \dots, 0) = f(z_1, \dots, z_n) \cdot \mathcal{D}.$

The composition $p \circ \iota = \mathrm{id}_{\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_n)[0]}$, and therefore

$$\widehat{\Psi}_{2k+1} = p^* \Psi_{2k+1} \tag{20}$$

satisfies the condition $\iota^* \widehat{\Psi}_{2k+1} = \Psi_{2k+1}$.

3.2. **The Integration in the Complex Case.** Let M be a complex manifold of dimension n and let $\Psi_{2n+1}, \Psi_{2n+3}, \Psi_{2n+5}, \dots$ be Lifting cocycles on the Lie algebra $\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_n)$ (see Section 1). We choose a holomorphic coordinate system $\varphi_x\colon U \to M$ ($U \subset \mathbb{C}^n$, $\varphi_x(0) = x$) in any point $x \in M$, depending smoothly on the point $x \in M$. Then the cocycle

$$\varphi_x^* \widehat{\Psi}_{2k+1} \in C^{2k+1}_{\mathrm{Lie}}(\Gamma_M(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_M) \otimes_{\mathcal{O}_M} \mathcal{D}^\bullet_M))$$

is defined in any point $x \in M$. As in Section 2, the cohomological class $[\varphi_x^*(\widehat{\Psi}_{2k+1})]$ does not depend on the point $x \in M$ and on the holomorphic coordinate system φ_x in the point x.

Analogously, let λ be a holomorphic line bundle on M, $p\colon \lambda \to M$ be the projection. Choosing a coordinate system $\varphi_{x,\lambda}\colon U \times \mathbb{C} \xrightarrow{\sim} p^{-1}(\varphi_x(U))$ in any point $x \in M$, smoothly depending on the point x, we define the cocycle

$$\varphi_{x,\lambda}^*(\widehat{\Psi}_{2k+1}) \in C^{2k+1}_{\mathrm{Lie}}(\Gamma_M(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\lambda,M}) \otimes_{\mathcal{O}_M} \mathcal{D}^\bullet_M)).$$

Again, the cohomological class $[\varphi_{x,\lambda}^*(\widehat{\Psi}_{2k+1})]$ does not depend on the point $x \in M$ and the coordinate system $\varphi_{x,\lambda}\colon U \times \mathbb{C} \xrightarrow{\sim} p^{-1}(\varphi_x(U))$.

Furthermore, we define the i-form Θ_i on M (as a C^∞-manifold) with values in

$$C^{2k+1-i}_{\mathrm{Lie}}(\Gamma_M(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\lambda,M}) \otimes_{\mathcal{O}_M} \mathcal{D}^\bullet_M))$$

by the formula:

(21)

$$\Theta_i(v_1, \dots, v_i)(\mathcal{D}_1, \dots, \mathcal{D}_{2k+1-i}) = \varphi_{x,\lambda}^*(\widehat{\Psi}_{2k+1})(\widetilde{t}_{v_1} \otimes \mathrm{Id}, \dots, \widetilde{t}_{v_i} \otimes \mathrm{Id}, \mathcal{D}_1, \dots, \mathcal{D}_{2k+1-i})$$

where:

(i) $x \in M$;

(ii) $v_1, \dots, v_i$ are tangent vectors to M (as a C^∞-manifold) in point x;

(iii) $\widetilde{t}_{v_1}, \dots, \widetilde{t}_{v_i}$ are differential operators of order ≤ 1 on M defined as in Subsec. 2.2 and $\widetilde{t}_{v_1} \otimes \mathrm{Id}, \dots, \widetilde{t}_{v_i} \otimes \mathrm{Id}$ are *infinite* matrices; strictly speaking, they do not lie in the algebra $\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\lambda,M}) \otimes_{\mathcal{O}_M} C^\infty_M$;

(iv)

$$\mathcal{D}_1, \dots, \mathcal{D}_{2k+1-i} \in \Gamma_M(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\lambda,M}) \otimes_{\mathcal{O}_M} C^\infty_M) = [\Gamma_M(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\lambda,M}) \otimes_{\mathcal{O}_M} \mathcal{D}^\bullet_M)]^0.$$

(v) cochain $\Theta_i(v_1, \dots, v_i)$ is equal to zero when one of the arguments has grading $\neq 0$.

Lemma. *The cocycles $\Theta_1, \dots, \Theta_{2n}$ satisfy the system* (9)*.*

Proof. The unique new point (after Theorem 2.2) is the existence of the component $\delta_{\mathrm{Lie}}^{\overline{\partial}}$ in the cochain differential δ_{Lie}, connected with the differential $\overline{\partial}$ in the DG Lie algebra $\Gamma_M(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\lambda,M}) \otimes_{\mathcal{O}_M} \mathcal{D}_M^\bullet)$. However, it follows from the definition (20) of the extended Lifting cocycles $\widehat{\Psi}_{2k+1}$ that $\delta_{\mathrm{Lie}}^{\overline{\partial}}(\Theta_i)(v_1, \ldots, v_i) \equiv 0$. ◻

The direct consequence of this Lemma is the statement that $\int_\sigma \Theta_i$ is a *cocycle* in

$$C_{\mathrm{Lie}}^{2k+1-i}(\Gamma_M(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\lambda,M}) \otimes_{\mathcal{O}_M} \mathcal{D}_M^\bullet))$$

for any singular i-cycle σ:

$$\delta_{\mathrm{Lie}} \int_\sigma \Theta_i = \int_\sigma \delta_{\mathrm{Lie}} \Theta_i = \int_\sigma d_{\mathrm{DR}} \Theta_{i-1} = 0.$$

One can prove also the direct analogs of Theorem 2.3 and Corollary 2.3.

3.3. **Holomorphic Noncommutative Residue.** Let M be a *compact* complex manifold, $\dim M = n$, $\langle M \rangle$ be the fundamental class of M, $\langle M \rangle \in H_{2n}^{\mathrm{sing}}(M; C)$. Then, according to previous Subsection,

$$\int_{\langle M \rangle} \Theta_{2n} \in C_{\mathrm{Lie}}^1(\Gamma_M(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\lambda,M}) \otimes_{\mathcal{O}_M} \mathcal{D}_M^\bullet))$$

is a *cocycle.*

Let us denote

$$A^\bullet = \Gamma_M(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\lambda,M}) \otimes_{\mathcal{O}_M} \mathcal{D}_M^\bullet).$$

Lemma. *Cocycle* $\int_{\langle M \rangle} \Theta_{2n}$ *defines a linear functional on*

$$\mathrm{Ker}\{\, \overline{\partial} \colon A_0/[A_0, A_0] \to A_1/[A_0, A_1] \,\}.$$

Proof. It follows directly from the generalization of the Theorem from [T] which states that

$$H_{\mathrm{Lie}}^\bullet(\mathfrak{gl}_\infty^{\mathrm{fin}}(A); C) \simeq S_{\mathrm{super}}^\bullet((HC_\bullet(A)[1])^*)$$

in the case of a DG associative algebra A. ◻

When $\mathcal{D} \in \mathrm{Ker}\{\, \overline{\partial} \colon A_0/[A_0, A_0] \to A_1/[A_0, A_1] \,\}$ we denote by $\mathrm{Tr}_\lambda(\mathcal{D})$ this noncommutative residue; in particular, $\mathrm{Tr}_\lambda(\mathbf{1}_\lambda)$ is well-defined, where $\mathbf{1}_\lambda$ is the identity differential operator in the line bundle λ.

We have proved (Corollary 2.3) that the number $\mathrm{Tr}_\lambda(\mathbf{1}_\lambda)$ does not depend on the choice of the coordinate systems in the definition of $\int_{\langle M \rangle} \Theta_{2n}$; therefore, this number is an *invariant* of the line bundle λ.

Conjecture. $\mathrm{Tr}_\lambda(\mathbf{1}_\lambda) = C(M) \cdot \chi(\lambda)$ *where* $C(M)$ *does not depend on* λ *and* $\chi(\lambda)$ *is the Euler characteristic of* λ.

This Conjecture is based on the discussions with Boris Feigin.

4. Computation in the case $M = \mathbb{C}P^n$

Unfortunately, the author has not found a simple computation of the integrals $\int_{\langle \mathbb{C}P^n\rangle} \Theta_{2n}$ for projective spaces by the method of the Dolbeault complex, described in Section 3. On the other hand, the computation using the Čech complex turns out to be relatively simple. Therefore, our viewpoint is a compromise: we don't develop the general theory of the integration by the method of the Čech complex, but in the case of the projective spaces we reprove the analogs of Theorem 2.3 and Corollary 2.3 for the Čech method.

The higher cohomology of the sheaf of holomorphic differential operators on $\mathbb{C}P^n$ vanishes, and we work with algebra of global holomorphic differential operators.

4.1. **The explicit construction of the integral.** We will work with the covering

$$U_1 \cup U_2 \cup \ldots \cup U_{n+1} = \mathbb{C}P^n = \{ (z_1, \ldots, z_{n+1})/\sim \mid z_i \in \mathbb{C}, \quad (i = 1, \ldots, n+1), \quad \text{not all } z_i \text{ are equal to } 0 \}.$$

where

$$U_i = \{ (z_1, \ldots, z_{n+1}) \mid z_i \neq 0 \}$$

For all $i = 1, \ldots, n+1$ and all the points $z^0 \in U_i$ we choose a holomorphic coordinate system $\iota_{z^0,U_i} \colon U_i \to \mathbb{C}^n$ ($\iota_{z^0,U_i}(z^0) = 0$), depending holomorphically on the point z^0 (for fixed U_i).

We set

(22)
$$\iota_{z^0,U_i}(z_1, \ldots, z_{n+1}) = \left(\frac{z_1}{z_i} - \frac{z_1^0}{z_i^0}, \ldots, \frac{z_{i-1}}{z_i} - \frac{z_{i-1}^0}{z_i^0}, \frac{z_{i+1}}{z_i} - \frac{z_{i+1}^0}{z_i^0}, \ldots, \frac{z_{n+1}}{z_i} - \frac{z_{n+1}^0}{z_i^0} \right).$$

We define the cocycles

$$\iota^*_{z^0,U_{j_1}}(\xi), \ldots, \iota^*_{z^0,U_{jk}}(\xi) \in C^l_{\mathrm{Lie}}(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_{\mathbb{C}P^n}); \mathbb{C})$$

for any cocycle $\xi \in C^l_{\mathrm{Lie}}(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_n); \mathbb{C})$ and any point $z^0 \in U_{j_1} \cap \ldots \cap U_{j_k}$.

In our choice, $\iota_{z^1,U_j} = \iota_{z^0,U_i} \circ A$ where A is a global holomorphic automorphism of $\mathbb{C}P^n$, $A \in GL_{n+1}(\mathbb{C})/\mathbb{C}^*$, for any i, j and $z^0 \in U_i$, $z^1 \in U_j$.

Construction of the integral:

I. For any $z^0 \in U_1 \cap U_i$ $(i > 1)$ we find a cochain

$$\Theta_{1,i}(z^0) \in C^{l-1}_{\mathrm{Lie}}(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_{\mathbb{C}P^n}); \mathbb{C})$$

such that

$$\iota^*_{z^0,U_i}\xi - \iota^*_{z^0,U_1}\xi = \delta_{\mathrm{Lie}}\Theta_{1,i}(z^0). \tag{23}$$

Furthermore, for $z_0 \in U_1 \cap U_2 \cap U_i$ $(i > 2)$ we find a cochain

$$\Theta_{1,2,i}(z^0) \in C^{l-2}_{\mathrm{Lie}}(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_{\mathbb{C}P^n}); \mathbb{C})$$

such that

$$\Theta_{1,i}(z^0) - \Theta_{1,2}(z^0) = \delta_{\mathrm{Lie}}\Theta_{1,2,i}(z^0) \tag{24}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Finally, for $z^0 \in U_1 \cap \ldots \cap U_{n+1}$ we find

$$\Theta_{1,2,\ldots,n+1}(z^0) \in C^{l-n}_{\mathrm{Lie}}(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_{\mathbb{C}P^n}); \mathbb{C})$$

such that

$$\Theta_{1,2,\dots,n-1,n+1}(z^0) - \Theta_{1,2,\dots,n-1,n}(z^0) = \delta\Theta_{1,2,\dots,n+1}(z^0). \tag{25}$$

II. For any point $z_0 \in U_1 \cap U_2 \cap \dots \cap U_{n+1}$ we find a solution $\{\Theta^1_{1,2,\dots,n+1}, \Theta^2_{1,2,\dots,n+1}, \dots, \Theta^n_{1,2,\dots,n+1}\}$,

$$\Theta^i_{1,2,\dots,n+1} \in C^{l-n-i}_{\text{Lie}}(\mathfrak{gl}^{\text{fin}}_\infty(\text{Dif}_{\mathbb{C}P^n}); \mathbb{C}) \otimes_{\mathbb{C}} \Omega^i_{U_1\cap\dots\cap U_{n+1}}$$

of the following system:

$$\begin{cases} d_{\text{DR}}\Theta_{1,2,\dots,n+1} = \delta_{\text{Lie}}\Theta^1_{1,2,\dots,n+1} \\ d_{\text{DR}}\Theta^1_{1,2,\dots,n+1} = \delta_{\text{Lie}}\Theta^2_{1,2,\dots,n+1} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ d_{\text{DR}}\Theta^{n-1}_{1,2,\dots,n+1} = \delta_{\text{Lie}}\Theta^n_{1,2,\dots,n+1} \end{cases} \tag{26}$$

(here we denote by $\Omega^i_{U_1\cap\dots\cap U_{n+1}}$ the space of *holomorphic* i-forms on $U_1 \cap \dots \cap U_{n+1}$).

We set $T^n = \{\, z_1 = 1,\ |z_2| = \dots = |z_{n+1}| = 1 \,\} \subset \mathbb{C}P^n$. We want to find conditions for which the following two statements hold:

(i) $\int_{T_n} \Theta^n_{1,2,\dots,n+1} \in C^{l-2n}_{\text{Lie}}(\mathfrak{gl}^{\text{fin}}_\infty(\text{Dif}_{\mathbb{C}P^n}); \mathbb{C})$ is a *cocycle*;

(ii) if $[\xi] = 0$ then $\left[\int_{T^n} \Theta^n_{1,2,\dots,n}\right] = 0$ (here $[\dots]$ denotes the cohomological class of the cocycle).

We find these conditions and prove both statements in the next Subsection.

4.2. **Main Theorem.** The following reformulation of the construction of the integral due on the previous Subsection will be very useful in the course of the proof of statements (i) and (ii).

Let $\sigma^i = \{\, (x_1, \dots, x_{i+1}) \in \mathbb{R}^{i+1} \mid \sum x_1 + x_2 + \dots + x_{i+1} = 1 \,\}$ be the i-simplex, we denote by $[1], [2], \dots, [i+1]$ its vertices, by $[1,2]$, $[1,3]$, $\dots$ — its 1-faces, and so on.

Let us define the space $\widetilde{\mathbb{C}P}^n$ by the following way. The space $\widetilde{\mathbb{C}P}^n$ is glued from the following pieces:

$$\begin{aligned} &U_1, \dots, U_{n+1}; \\ &(U_i \cap U_j) \times \sigma^1 \qquad (i < j); \\ &(U_i \cap U_j \cap U_k) \times \sigma^2 \qquad (i < j < k); \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &(U_1 \cap U_2 \cap \dots \cap U_{n+1}) \times \sigma^n. \end{aligned}$$

We glue to the space

$$(U_1 \cap \dots \cap U_{n+1}) \times [1, 2, \dots, n] \subset (U_1 \cap \dots \cap U_{n+1}) \times \sigma^n$$

(here $[1, 2, \dots, n]$ is a $(n-1)$-face of the simplex σ^n) the "piece" $(U_1 \cap \dots \cap U_n) \times \sigma^{n-1}$, to the space $(U_1 \cap \dots \cap U_{n+1}) \times [1, 2, \dots, n-1, n+1]$ the "piece" $(U_1 \cap \dots \cap U_{n-1} \cap U_{n+1}) \times \sigma^{n-1}$, and so on. We glue all the pieces to each other along the faces of simplexes, and after that we obtain a *connected* space, we denote it by $\widetilde{\mathbb{C}P}^n$.

The projection $p\colon \widetilde{\mathbb{C}P}^n \to \mathbb{C}P^n$ is well-defined (the fibers are simplexes of different dimension).

We connect a holomorphic coordinate system in the neighbourhood of the point $p(\widetilde{z})$ with any point $\widetilde{z} \in \mathbb{C}P^n$ in the following way.

First of all, let us suppose, that $\widetilde{z} \in (U_1 \cap \ldots \cap U_{n+1}) \times \sigma^n$, $z^0 = p(\widetilde{z})$. The coordinate systems, connected with points $z^0 \times [1]$, $z^0 \times [2]$, $\ldots$, $z^0 \times [n+1]$ are the coordinate systems $\iota_{z^0,U_1}, \ldots, \iota_{z^0,U_{n+1}}$ correspondently.

Furthermore, we define a coordinate systems on whole simplex $z^0 \times \sigma^n$ such that the following conditions (1)–(4) hold:

(1) the coordinate system depends smoothly on the point of σ^n;

(2) the coordinate systems, connected with the points $z^0 \times [1]$, $\ldots$, $z^0 \times [n+1]$, are $\iota_{z^0,U_1}, \ldots, \iota_{z^0,U_{n+1}}$;

(3) for fixed $t \in \sigma^n$, the coordinate system in the point $z^0 \times t$ depends *holomorphically* on z^0;

(4) if $t \in [\iota_1, \ldots, \iota_k] \subset \sigma^n$, then coordinate system on $(U_1 \cap \ldots \cap U_{n+1}) \times t$ can be extended to a *holomorphic* coordinate system on $(U_{i_1} \cap \ldots \cap U_{i_k}) \times t$.

Let us suppose, that conditions (1)–(4) hold. Then one can associate to any point $\widetilde{z}$ a holomorphic coordinate system in the neighbourhood of the point $p(\widetilde{z})$. Therefore, for a fixed cocycle $\xi \in C^l_{\mathrm{Lie}}(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_n);\mathbb{C})$ one can associate the cocycle $\xi(\widetilde{z}) \in C^l_{\mathrm{Lie}}(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_{\mathbb{C}P^n});\mathbb{C})$.

We consider the *canonical* solution $\{\Theta_1, \ldots, \Theta_{2n}\}$, $\Theta_i \in \Omega^i_{\widetilde{\mathbb{C}P}^n} \otimes C^l_{\mathrm{Lie}}(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_{\mathbb{C}P^n});\mathbb{C})$, of the system:

$$\begin{cases} d_{\mathrm{DR}}\xi(\widetilde{z}) = \delta_{\mathrm{Lie}}\Theta_1 \\ d_{\mathrm{DR}}\Theta_1 = \delta_{\mathrm{Lie}}\Theta_2 \\ \dots\dots\dots\dots\dots\dots \\ d_{\mathrm{DR}}\Theta_{2n-1} = \delta_{\mathrm{Lie}}\Theta_{2n} \end{cases} \tag{27}$$

(see formula (14)).

The $2n$-form Θ_{2n} on $\widetilde{\mathbb{C}P}^n$ depends *holomorphically* on $p(\widetilde{z})$, therefore, $\Theta_{2n} = 0$ for $\widetilde{z} \notin (U_1 \cap \ldots \cap U_{n+1}) \times \sigma^n$.

Theorem. *If conditions* (1)–(4) *above hold, then*:

(i) $\int_{T^n \times \sigma^n} \Theta_{2n} \in C^{l-2n}_{\mathrm{Lie}}(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_{\mathbb{C}P^n});\mathbb{C})$ *is a cocycle*;

(ii) *if* $[\xi] = 0$ *then* $\left[\int_{T^n \times \sigma^n} \Theta_{2n}\right] = 0$.

(Here $T^n \subset U_1 \cap \ldots \cap U_{n+1}$, $T^n = \{(1, z_2, \ldots, z_{n+1}) \mid |z_2| = \ldots = |z_{n+1}| = 1.\}$)

Proof. (ii) is a direct consequence of (i): one can repeat the arguments of Theorem 2.3.(1). Let us prove (i).

We have:

$$\delta_{\mathrm{Lie}}\left(\int_{T^n \times \sigma^n} \Theta_{2n}\right) = \int_{T^n \times \sigma^n} \delta_{\mathrm{Lie}}\Theta_{2n} = \int_{T^n \times \sigma^n} d_{\mathrm{DR}}\Theta_{2n-1} = \int_{T^n \times \partial(\sigma^n)} \Theta_{2n-1}.$$

If a point t lies in a $(n-1)$-face of the simplex σ^n, say $t \in [1, 2, \ldots, n]$, then one can extended Θ_{2n-1} from $(U_1 \cap \ldots \cap U_{n+1}) \times [1, 2, \ldots, n]$ to a $(2n-1)$-form on $(U_1 \cap \ldots \cap U_n) \times [1, 2, \ldots, n]$, which depends *holomorphically* on the first argument. Therefore, the $(2n-1)$-form Θ_{2n-1} is defined on $(S^1 \times \ldots \times S^1 \times D^2) \times [1, 2, \ldots, n]$ and depends *holomorphically* on the point of $z \in D^2$ via condition (4) (here $D^2 = \{\, z \in \mathbb{C} \mid |z| \leq 1 \,\}$, $\partial D^2 = S^1$). Therefore $\int_{T^n \times [1,2,\ldots,n]} \Theta_{2n-1} = 0$ according to the Cauchy Theorem. The proof is the same for other $(n-1)$-faces in $\partial\sigma^n$. ∎

In the notations of Subsec. 4.1 we have:

$$\int_{z^0\times[1,i]}\Theta_1=\Theta_{1,i}(z_0),$$
$$\int_{z^0\times[1,2,i]}\Theta_2=\Theta_{1,2,i}(z^0),$$
$$\dots\dots\dots\dots\dots\dots$$
$$\int_{z^0\times[1,2,\dots,n+1]}\Theta_n=\Theta_{1,2,\dots,n}(z_0).$$

Therefore, the definition of the integral, given in this Subsection, coincides with the definition, given in Subsec. 4.1.

Remark. The Theorem remains true for the Lie algebra $\mathfrak{gl}^{\mathrm{fin}}_{\infty}(\mathrm{Dif}_{\lambda,\mathbb{C}P^n})$ for any holomorphic line bundle λ on $\mathbb{C}P^n$. The proof is the same.

Lemma. *The coordinate systems $\iota_{z^0,U_1},\dots,\iota_{z^0,U_{n+1}}$, defined by the formula* (22)*, satisfy conditions* (1)–(4) *above.*

Proof. Straightforward. □

4.3. **Computation for $\mathbb{C}P^1$.**

4.3.1. We denote U_1 by V and U_2 by U, and let $z^0=(x_0,y_0)$ be a point $U\cap V\subset\mathbb{C}P^1$. Let us find $\Theta_{1,2}(x_0,y_0)$ (see Subsection 4.1).

If $\iota_{z^0,U},\iota_{z^0,V}$ are the coordinate systems, defined by (22), then the maps $j_{z^0,U},j_{z^0,V}\colon\mathrm{Dif}_{\mathbb{C}P^1}\to\mathrm{Dif}_1$ are defined by the formulas:

$$\begin{aligned} j_{z^0,U}(\mathcal{D})(f)&=(\iota^{-1}_{z^0,U})^*\mathcal{D}(\iota^*_{z^0,U}f),\\ j_{z^0,V}(\mathcal{D})(f)&=(\iota^{-1}_{z^0,V})^*\mathcal{D}(\iota^*_{z^0,V}f).\end{aligned}\tag{28}$$

We have: $\iota_{z^0,V}=\iota_{z^0,U}\circ A_{z^0}$, where $A\in GL_2(\mathbb{C})/\mathbb{C}^*$ is the global automorphism of $\mathbb{C}P^1$, defined by the matrix:

$$A_{z^0}=\begin{pmatrix}\frac{x_0^2}{y_0^2}-1 & \frac{x_0}{y_0}\\ \frac{x_0}{y_0} & 0\end{pmatrix}.$$

From (28) we have:

$$j_{z^0,V}(\mathcal{D})=j_{z^0,U}(A^{-1}_{z^0}\mathcal{D}A^{-1}_{z^0})\tag{29}$$

Here the differential operator $A^{-1}_{z^0}\mathcal{D}A_{z^0}$ defined by a differential operator $\mathcal{D}$ by the formulas: $A^{-1}_{z^0}\mathcal{D}A_{z^0}(f)=\widetilde{f}$, where

$$\widetilde{f}(x)=\widehat{f}(A^{-1}_{z^0}x)\quad\text{and}\quad\widehat{f}(x)=\mathcal{D}(f\circ A_{z^0}(x)).\tag{30}$$

Denote $A^{-1}\mathcal{D}A$ by $\mathrm{Ad}(A^{-1})(\mathcal{D})$.

We have:

$$\iota^*_{z^0,U}\xi-\iota^*_{z^0,V}\xi=\iota^*_{z^0,U}\xi-\mathrm{Ad}(A^{-1}_{z^0})\cdot\iota^*_{z^0,U}\xi.$$

4.3.2. Let A_t be a matrix *piece-wise* smooth path, $A_t \in GL_n(\mathbb{C})$, $t \in [0,1]$. We have:

$$(\mathrm{Ad}(A_1) - \mathrm{Ad}(A_0))(x) = \sum_{\dots}(A_{t+\varepsilon}xA_{t+\varepsilon}^{-1} - A_txA_t^{-1}) = \\ = \sum_{\dots}(\mathrm{Ad}(A_{t+\varepsilon}A_t^{-1}) - 1)(A_txA_t^{-1}) = \int_{t\in[0,1]} \mathrm{ad}(a_t)(A_txA_t^{-1}) \tag{31}$$

where

$$a_t = \frac{d}{d\varepsilon}(A_{t+\varepsilon}A_t^{-1})\big|_{\varepsilon=0}. \tag{32}$$

Let us choose a path, connecting Id and A_{z^0}.
Suppose:

$$X_t = \begin{pmatrix} e^{\pi it} & 0 \\ 0 & 1 \end{pmatrix}, \qquad t \in [0,1],$$

$$Y_t = \begin{pmatrix} -\cos\frac{\pi}{2}t & \sin\frac{\pi}{2}t \\ \sin\frac{\pi}{2}t & \cos\frac{\pi}{2}t \end{pmatrix}, \qquad t \in [0,1],$$

$$Z_t = \begin{pmatrix} t^2-1 & t \\ t & 0 \end{pmatrix}, \qquad t \text{ be a complex path, connecting } 1 \text{ and } \frac{x_0}{y_0}.$$

We have:

$$X_0 = \mathrm{Id}, \quad X_1 = Y_0, \quad Y_1 = Z_1, \quad Z_{\frac{x_0}{y_0}} = A_{z^0}.$$

According to the formulas (31) and (32), we have:

$$\iota^*_{z^0,U}\xi - \iota^*_{z^0,V}\xi = -\int_0^1 \mathrm{ad}(x_t)(\mathrm{Ad}(X_t)^{-1}(\iota^*_{z^0,U}\xi)) - \\ -\int_0^1 \mathrm{ad}(y_t)(\mathrm{Ad}(Y_t^{-1})(\iota^*_{z^0,U}\xi)) - \int_1^{x_0/y_0} \mathrm{ad}(z_t)(\mathrm{Ad}(Z_t^{-1})(\iota^*_{z^0,U}\xi)). \tag{33}$$

In the affine coordinate $z = \frac{x}{y}$ on U we have:

$$x_t = -\pi iz\frac{\partial}{\partial z},$$
$$y_t = \frac{\pi}{2}z\frac{\partial}{\partial z},$$
$$z_t = -\left(1+\frac{1}{t^2}\right)z^2\frac{\partial}{\partial z}.$$

Let us define the $(l-1)$-cochain

$$\varphi_X(\mathcal{D}_1,\dots,\mathcal{D}_{l-1}) = \xi\left(j_{z^0,U}(X_t^{-1}x_tX_t), j_{z^0,U}(X_t^{-1}\mathcal{D}_1X_t),\dots,j_U(X_t^{-1}\mathcal{D}_{l-1}X_t)\right). \tag{34}$$

Then

$$\mathrm{ad}(x_t)(\mathrm{Ad}(X_t^{-1})(\iota^*_{z^0,U}))(\mathcal{D}_1,\dots,\mathcal{D}_{l-1}) = (\delta_{\mathrm{Lie}}\varphi_X)(\mathcal{D}_1,\dots,\mathcal{D}_{l-1}). \tag{35}$$

There are analogous formulas for Y_t and Z_t.

As we will see below, the values of $\int_{S^1}\Theta^1_{1,2}$ depend only on the Z_t-term.

4.3.3. Denote $S^1 = \left\{ (x,y) \in \mathbb{C}P^1 \,\middle|\, \left|\frac{x}{y}\right| = 1 \right\}$. We consider the coordinate systems $\iota_{z^0,U}$ in all the points $z^0 \in S^1$. The (global) holomorphic vector field on $\mathbb{C}P^1$, corresponding to a tangent vector to S^1 in any point (in the sense of Subsec. 2.1) is equal, up to a constant factor, $\frac{\partial}{\partial z}$ (in the affine coordinate in U).

Therefore, the X_t- and Y_t-terms in $\Theta^1_{1,2}$ have, up to a constant factor, the following form:

$$\int \mathrm{Ad}(X_t^{-1})\xi \left(j_{z^0,U}\left(z\frac{\partial}{\partial z}\right), j_{z^0,U}\left(\frac{\partial}{\partial z}\right), j_{z^0,U}(\mathcal{D}_1), \dots, j_{z^0,U}(\mathcal{D}_{l-2}) \right)$$

and

$$\int \mathrm{Ad}(Y_t^{-1})\xi \left(j_{z^0,U}\left(z\frac{\partial}{\partial z}\right), j_{z^0,U}\left(\frac{\partial}{\partial z}\right), j_{z^0,U}(\mathcal{D}_1), \dots, j_{z^0,U}(\mathcal{D}_{l-2}) \right).$$

These terms have nonzero grading as expression of the form $\Psi(\mathcal{D}_1, \dots, \mathcal{D}_{l-2})$, and therefore the sequel computations do not depend on these terms. Z_t-term has a form

$$\int \mathrm{Ad}(Z_t^{-1})\xi \left(j_{z^0,U}\left(z^2\frac{\partial}{\partial z}\right), j_{z^0,U}\left(\frac{\partial}{\partial z}\right), j_{z^0,U}(\mathcal{D}_1), \dots, j_{z^0,U}(\mathcal{D}_{l-2}) \right)$$

and this expression has the grading 0.

Let us calculate Z_t-term. We have:

$$(Z_t) = \int_{\frac{x_0}{y_0} \in S^1} \int^{\varphi_0} \mathrm{Ad}(Z_t^{-1})\xi \left(j_{z^0,U}\left(\frac{\partial}{\partial z}\right), j_{z^0,U}(Z_t), j_{z^0,U}(\mathcal{D}_1), \dots, j_{z^0,U}(\mathcal{D}_{l-2}) \right)$$

here $\frac{x_0}{y_0} = e^{2\pi\varphi_0}$, $\varphi_0 \in [0,1]$, $Z_t = -\left(1 + \frac{1}{t^2}\right) z^2 \frac{\partial}{\partial z}$.

Calculation of the integral:

$$(Z_t) = \int_{\varphi_0 \in [0,1]} \int_0^{\varphi_0} \left(1 + \frac{1}{e^{4\pi i\beta}}\right) \cdot \left(\frac{\partial}{\partial z}, z^2\frac{\partial}{\partial z}, j_{z^0,U}(\mathrm{Ad}(Z_t^{-1})(\mathcal{D}_1)), \dots, j_{z^0,U}(\mathrm{Ad}(Z_t^{-1})(\mathcal{D}_{l-2})) \right) d\beta d\varphi_0.$$

(here $t = e^{2\pi i\beta}$).

4.3.4. Let $\xi = \Psi_{2k+1}$ ($k \geq 1$) be the Lifting cocycle on the Lie algebra $\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_1)$ (see Subsec. 1.2), and $\mathcal{D}_1, \dots, \mathcal{D}_{2k-1} \in \mathfrak{gl}^{\mathrm{fin}}_\infty \otimes 1$. Then $\mathrm{Ad}(Z_t^{-1})(\mathcal{D}_i) = \mathcal{D}_i$, and we are able to conclude the calculation. We have:

$$(Z_t) = \left(\frac{1}{2} + \frac{1}{4\pi i}\right) \Psi_{2k+1}\left(\frac{\partial}{\partial z}, z^2\frac{\partial}{\partial z}, \mathcal{D}_1, \dots, \mathcal{D}_{2k-1}\right) \tag{36}$$

where $\mathcal{D}_i \in \mathfrak{gl}^{\mathrm{fin}}_\infty \otimes 1$.

It follows directly from our definitions that $\frac{\partial}{\partial z}$ *and* $z^2\frac{\partial}{\partial z}$ *in* (36) *are infinite matrices* $\mathrm{Id} \otimes \frac{\partial}{\partial z}$ *and* $\mathrm{Id} \otimes z^2\frac{\partial}{\partial z}$. *Strictly speaking, these matrices do not lie in the Lie algebra* $\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_1)$ (*see also* (21)).

Theorem. *The cochain* $\Psi_{2k+1}\left(\frac{\partial}{\partial z}, z^2\frac{\partial}{\partial z}, \mathcal{D}_1 \dots, \mathcal{D}_{2k-1}\right)$ *has nonzero value on some* $(2k-1)$*-cycle on the Lie algebra* $\mathfrak{gl}^{\mathrm{fin}}_\infty \otimes 1$.

Proof. First of all we consider the simplest case $k=1$. We have to calculate $\Psi_3(\partial, x^2\partial, 1)$ (here $\partial = \partial \otimes \mathrm{Id}$, $x^2\partial = x^2\partial \otimes \mathrm{Id}$, $1 = E_{11}$. We have:

$$\Psi_3(A_1, A_2, A_3) = \operatorname*{Alt}_{A,D} \mathrm{Tr}(D_1A_1 \cdot D_2A_2 \cdot A_3) + \operatorname*{Alt}_{A} \mathrm{Tr}(Q \cdot A_1 \cdot A_2 \cdot A_3),$$

where $D_1 = \operatorname{ad} \ln \partial$, $D_2 = \operatorname{ad} \ln x$,

$$Q = x^{-1}\partial^{-1} + \frac{1}{2}x^{-2}\partial^{-2} + \ldots + \frac{(n-1)!}{n}x^{-n}\partial^{-n} + \ldots \qquad \text{(see Subsec. 1.2).}$$

The first summand in $\Psi_3(\partial, x^2\partial, 1)$ is equal to -2, the second one is equal to -1. Therefore, $\Psi_3(\partial, x^2\partial, 1) = -3 \neq 0$.

The general case:

let $\sum\limits_i A_1^{(i)} \wedge \cdots \wedge A_{2k-1}^{(i)}$ be a cycle in $\mathfrak{gl}_\infty^{\mathrm{fin}} \otimes 1$, we have to calculate

$$\sum_i \Psi_{2k+1}(\partial, x^2\partial, A_1^{(i)}, \ldots, A_{2k-1}^{(i)}) \qquad \text{(see Subsec. 1.2).}$$

The "leading" term $\operatorname*{Alt}\limits_{A,D} \mathrm{Tr}(D_1A_1 \cdot D_2A_2 \cdot A_3 \cdot \ldots \cdot A_{2k+1})$ is equal to $-2\mathrm{Tr}\left(\sum\limits_i \mathrm{Alt}(A_1^{(i)} \cdot A_2^{(i)} \cdot \ldots \cdot A_{2k-1}^{(i)})\right)$. Furthermore, we consider the term

$$\operatorname*{Alt}_{A,D}(D_1A_1 \cdot A_2 \cdot \ldots \cdot A_{2s-1} \cdot D_2A_{2s} \cdot A_{2s+1} \cdot \ldots \cdot A_{2k+1}).$$

It is necessarily that $A_1 = x^2\partial \otimes \mathrm{Id}$, $A_{2s} = \partial \otimes \mathrm{Id}$ (otherwise this expression is equal to 0), and matrices $D_1(x^2\partial \otimes \mathrm{Id})$ and $D_2(\partial \otimes \mathrm{Id})$ commute with $\mathfrak{gl}_\infty^{\mathrm{fin}} \otimes 1$. Therefore, every such term is equal to

$$-2\mathrm{Tr}\left(\sum_i \mathrm{Alt}(A_1^{(i)} \cdot \ldots \cdot A_{2k-1}^{(i)})\right).$$

The remaining term $\operatorname*{Alt}\limits_{A} \mathrm{Tr}(Q \cdot A_1 \cdot A_2 \cdot \ldots \cdot A_{2k+1})$ is equal to -1. Therefore,

$$\sum_i \Psi_{2k+1}(\partial \otimes \mathrm{Id}, x^2\partial \otimes \mathrm{Id}, A_1^{(i)}, \ldots, A_{2k-1}^{(i)}) =$$

$$= \frac{1}{2}(k+1) \cdot \left(-2\mathrm{Tr}\left(\sum_i \mathrm{Alt}(A_1^{(i)} \cdot \ldots \cdot A_{2k-1}^{(i)})\right)\right) - \mathrm{Tr}\left(\sum_i \mathrm{Alt}(A_1^{(i)} \cdot \ldots \cdot A_{2k-1}^{(i)})\right) =$$

$$= -(k+2) \cdot \mathrm{Tr}\left(\sum_i \mathrm{Alt}(A_1^{(i)} \cdot \ldots \cdot A_{2k-1}^{(i)})\right).$$

It follows from the standard facts on the cohomology of the Lie algebra $\mathfrak{gl}_\infty^{\mathrm{fin}}$ (see [F], Ch. 2.1) that the last expression is not equal to 0 for some cycle $\sum\limits_i A_1^{(i)} \wedge \cdots \wedge A_{2k-1}^{(i)}$. □

4.3.5. Using the same methods, one can show, that the value of

$$\int_{\mathbb{C}P^1} \Psi_{2k+1} \in C^{2k-1}_{\mathrm{Lie}}(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\lambda, \mathbb{C}P^1}); \mathbb{C})$$

on the matrix cycle $\sum_i A_1^{(i)} \wedge \dots \wedge A_{2k-1}^{(i)}$ (here $\lambda = \mathcal{O}(\boldsymbol{\lambda})$) is equal, up to a nonzero undepending on $\boldsymbol{\lambda}$ constant, to

$$\sum_i \Psi_{2k+1}(\mathrm{Id}\otimes\partial, \mathrm{Id}\otimes(x^2\partial - \boldsymbol{\lambda}x), A_1^{(i)}, \dots, A_{2k-1}^{(i)}) =$$

$$= -(k+2)(\boldsymbol{\lambda}+1)\cdot\left(\sum_i \mathrm{Tr}\,\mathrm{Alt}\, A_1^{(i)}\cdot\dots\cdot A_{2k-1}^{(i)}\right).$$

Let us note, that the number $\boldsymbol{\lambda}+1$ is equal to the Euler characteristic $\chi(\mathcal{O}(\boldsymbol{\lambda}))$.

4.3.6.

Theorem.

$$H^\bullet(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_1);\mathbb{C}) = \wedge^\bullet(\Psi_3, \Psi_5, \Psi_7, \dots).$$

Proof. It follows from Theorem 4.2 and Theorem 4.3.4 that the cocycles $\Psi_3, \Psi_5, \Psi_7, \dots$ are noncohomologous to zero cocycles on the Lie algebra $\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_1)$.

Furthermore, the standard *homomorphism of the Lie algebras*

$$\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_1)\oplus\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_1)\to\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_1)$$

defines the Hopf algebra structure on the cohomology $H^\bullet(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_1);\mathbb{C})$. We denote by Δ the coproduct in this Hopf algebra.

Lemma. *The Lifting cocycles* $\Psi_3, \Psi_5, \Psi_7, \dots$ *are primitive elements in* $H^\bullet(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_1);\mathbb{C})$, *i. e.* $\Delta\Psi_{2k+1} = 1\otimes\Psi_{2k+1} + \Psi_{2k+1}\otimes 1$.

Proof. It is obvious. □

It is well-known theorem, that any (super-) commutative and cocommutative Hopf algebra A is equal to (super-) symmetric algebra, generating by its primitive elements (i. e. such elements $a\in A$ that $\Delta a = 1\otimes a + a\otimes 1$). Therefore, there do not exist any nontrivial relations between $\Psi_3, \Psi_5, \Psi_7, \dots$ in $H^\bullet(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_1);\mathbb{C})$. On the other hand, according to [FT1] $H^\bullet(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_1);\mathbb{C})$ is the exterior algebra with the unique generator in each dimension $3, 5, 7\dots.$ It proves, that

$$H^\bullet(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_1);\mathbb{C}) = \wedge^\bullet(\Psi_3.\Psi_5, \Psi_7\dots).$$

□

4.3.7.

Theorem. *Let* $\iota\colon \mathrm{Dif}_{\mathbb{C}P^1}\to\mathrm{Dif}_1$ *is the map, connected with a choice of coordinate system in any point of* $\mathbb{C}P^1$. *Then*

$$H^\bullet(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\mathbb{C}P^1});\mathbb{C}) \simeq \wedge^\bullet\left(\int_{\mathbb{C}P^1}\Psi_3;\ \iota^*\Psi_3, \int_{\mathbb{C}P^1}\Psi_5;\ \iota^*\Psi_5, \int_{\mathbb{C}P^1}\Psi_7;\ \dots\right)$$

Proof. First of all, one can show by the method of [FT1], that $H^\bullet(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\mathbb{C}P^1});\mathbb{C})$ is the exterior algebra with the unique generator is dimension 1 and two generators in each dimension $3, 5, 7, \dots.$ Furthermore, there exists the standard map of the Lie algebras $\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\mathbb{C}P^1})\oplus\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\mathbb{C}P^1})\to\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\mathbb{C}P^1})$, which defines the Hopf algebra structure on cohomology $H^\bullet(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\mathbb{C}P^1});\mathbb{C})$. It is easy to prove that $\int_{\mathbb{C}P^1}\Psi_{2k+1}$ and $\iota^*\Psi_{2k+1}$ ($k\geq 1$) are *primitive* elements in $H^\bullet(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\mathbb{C}P^1});\mathbb{C})$ with respect to the Hopf algebra

structure. It follows from Theorem 4.3.4, that $\int_{\mathbb{C}P^1}\Psi_{2k+1}$ $(k\geq 1)$ are nonzero elements in $H^\bullet(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\mathbb{C}P^1});\mathbb{C})$. It remains to prove that $[\iota^*\Psi_{2k-1}]\neq 0$, $k\geq 2$, and that the cocycles $\iota^*\Psi_{2k-1}$ and $\int_{\mathbb{C}P^1}\Psi_{2k+1}$ are not cohomologous.

To prove the first statement, let us note, that the integral $\int_{\mathbb{C}P^1}\Psi_{2k-1}$ is defined via the pull-backs $\iota^*\Psi_{2k-1}$, connected with all the points of $\mathbb{C}P^1$. It follows from Lemma 2.1 that all these cocycles (connected to different points and a different choice of coordinate systems) are cohomologous (Strictly speaking, we have to use the special class of admissible coordinate systems, ... but is sufficient for the definition of the integral.). Therefore, it follows from Theorem 2.3 and Corollary 2.3 that if the cocycle $\iota^*\Psi_{2k-1}$ is cohomologous to 0 than $\int_{\mathbb{C}P^1}\Psi_{2k-1}$ is also cohomologous to zero, with the contradiction to Theorem 4.3.4.

To prove the second statement, let us note, that the value of $\int_{\mathbb{C}P^1}\Psi_{2k+1}$ on some *matrix* cycle is not equal to 0, but is obvious, that the value of $\iota^*\Psi_{2k-1}$ on any matrix cocycle is equal to 0. $\square$

It is true also, that for $\lambda=\mathcal{O}(\boldsymbol{\lambda})$, $\boldsymbol{\lambda}\neq -1$ and (in the sense of the algebras of *twisted* differential operators $\mathrm{Dif}_{\lambda,\mathbb{C}P^1}$, $\boldsymbol{\lambda}\in\mathbb{C}$), $6\left(\frac{\boldsymbol{\lambda}}{2}\right)^2+6\left(\frac{\boldsymbol{\lambda}}{2}\right)+1\neq 0$ than

$$H^\bullet(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\lambda,\mathbb{C}P^1});\mathbb{C})\simeq\wedge^\bullet\left(\int_{\mathbb{C}P^1}\Psi_3;\ \iota^*\Psi_3,\int_{\mathbb{C}P^1}\Psi_5;\ \iota^*\Psi_5,\int_{\mathbb{C}P^1}\Psi_7;\ \dots\right).$$

It is an interesting exercise to understand what appears when $\boldsymbol{\lambda}=-1$ or $6\left(\frac{\boldsymbol{\lambda}}{2}\right)^2+6\left(\frac{\boldsymbol{\lambda}}{2}\right)+1=0$.

4.4. **Computation for $\mathbb{C}P^n$.** Let us denote by $\mathrm{Dif}_{\lambda,\mathbb{C}P^n}$ the associative algebra of global holomorphic differential operators in the bundle $\mathcal{O}(\boldsymbol{\lambda})$ on $\mathbb{C}P^n$.

Using the methods, analogous to methods of Subsec. 4.3, one can prove, that the value of the integral

$$\int_{\mathbb{C}P^n}\Psi_{2k+1}\in C^{2k-2n+1}_{\mathrm{Lie}}(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\lambda,\mathbb{C}P^n});\mathbb{C})\qquad(k\geq n)$$

on a *matrix* cycle $\sum_i A_1^{(i)}\wedge\dots\wedge A^{(i)}_{2k-2n+1}$ $(A_j^{(i)}\in\mathfrak{gl}_\infty^{\mathrm{fin}}\otimes 1)$ is equal to

$$\sum_i\Psi_{2k+1}\left(\partial_1,\dots,\partial_n,x_1\left(\sum_{j=1}^n x_j\partial_j\right)-\boldsymbol{\lambda}x_1,\dots,x_n\left(\sum_j x_j\partial_j\right)-\boldsymbol{\lambda}x_n,\right.$$
$$\left.A_1^{(i)},\dots,A^{(i)}_{2k-2n+1}\right).\tag{37}$$

In (37) ∂_i and $x_i\left(\sum_{j=1}^n x_j\partial_j\right)-\boldsymbol{\lambda}x_i$ denotes $\mathrm{Id}\otimes\partial_i$ and $\mathrm{Id}\otimes\left(x_i\left(\sum_{j=1}^n x_j\partial_j\right)-\boldsymbol{\lambda}x_i\right)$ in particular, these are *infinite* matrices. According to Remark 4.2, it is sufficient to prove that $\int_{\mathbb{C}P^n}\Psi_{2k+1}\in C^{2k-2n+1}_{\mathrm{Lie}}(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_{\lambda,\mathbb{C}P^n});\mathbb{C})$ is not cohomologous to 0 in order to prove that $\Psi_{2k+1}\in C^{2k+1}_{\mathrm{Lie}}(\mathfrak{gl}_\infty^{\mathrm{fin}}(\mathrm{Dif}_n);\mathbb{C})$ is not cohomologous to 0 $(k\geq n)$.

The difference between the general case of $\mathbb{C}P^n$, $n\geq 1$ and the case of $\mathbb{C}P^1$ is that it is difficult to compute the whole polynomial on $\boldsymbol{\lambda}$, arising in the formula (37), or even

its value for $\boldsymbol{\lambda}=0$ (Conjecture 3.3 gives us the expected answer.) It is sufficient for our aims to prove that this polynomial is not equal to 0; therefore, it is sufficient to prove that its *leading coefficient* is not equal to 0. For the calculation of the leading coefficient on $\boldsymbol{\lambda}$ in (37), it is sufficient to calculate

$$\sum_i \Psi_{2k+1}(\partial_1,\dots,\partial_n,\boldsymbol{\lambda}x_1,\dots,\boldsymbol{\lambda}x_n,A_1^{(i)},\dots,A_{2k-2n+1}) = \boldsymbol{\lambda}^n \sum_i \Psi_{2k+1}(\partial_1,\dots,\partial_n,x_1,\dots,x_n,A_1^{(i)},\dots,A_{2k-2n+1}^{(i)}). \tag{38}$$

It is sufficient to find a matrix $(2k-2n+1)$-cycle $\sum_i A_1^{(i)}\wedge\dots\wedge A_{2k-2n+1}^{(i)}$ such that

$$\sum_i \Psi_{2k+1}(\partial_1,\dots,\partial_n,x_1,\dots,x_n,A_1^{(i)},\dots,A_{2k-2n+1}^{(i)}) \neq 0$$

(here ∂_i and x_i denotes $\mathrm{Id}\otimes\partial_i$ and $\mathrm{Id}\otimes x_i$ respectively) $(k\geq n)$.

First of all, let us consider the simplest case $k=n$. The leading term in $\Psi_{2n+1}(A_1,\dots,A_{2n+1})$ is equal to

$$\operatorname*{Alt}_{A,D}(D_1A_1\cdot\ldots\cdot D_{2n}A_{2n}\cdot A_{2n+1}) \qquad \text{(see Subsec. 1.3).} \tag{39}$$

We set:

$$D_1=\operatorname{ad}\ln\partial_1,\quad D_2=\operatorname{ad}\ln x_1,\quad D_3=\operatorname{ad}\ln\partial_2,\quad D_4=\operatorname{ad}\ln x_2,\ \dots,$$
$$A_1=x_1,\quad A_2=\partial_1,\quad A_3=x_2,\quad A_4=\partial_2,\ \dots,\quad A_{2n+1}=1.$$

Then the leading term (39) is equal to $(-1)^n\cdot(2n)!$. Furthermore, let us consider arbitrary term in $\Psi_{2n+1}(A_1,\dots,A_{2n+1})$ (connected with the interval of length $2n-2$ with marked points — see Subsec. 1.3). We may remove all the blocks of the form $A_i\cdot Q_{j,j+1}\cdot A_{i+1}$ in the left-hand side of the expression, its value will not change. Thus, we have the expression:

$$\operatorname*{Alt}_{A,D}\operatorname{Tr}(A_1\cdot Q_{12}\cdot A_2\cdot A_3\cdot Q_{34}\cdot A_4\cdot A_5\cdot Q_{56}\cdot A_6\cdot\ldots\cdot A_{2l-1}\cdot Q_{2l-1,2l}\cdot A_{2l}\cdot D_{2l+1}A_{2l+1}\cdot\ldots\cdot D_{2n}A_{2n}\cdot A_{2n+1}) \tag{40}$$

(a sign does not appear from such a permutation). We have:

$$(40)=\operatorname*{Alt}_{A,D}\operatorname{Tr}(A_1\cdot A_2\cdot Q_{12}\cdot A_3\cdot A_4\cdot Q_{34}\cdot\ldots\cdot A_{2l-1}\cdot A_{2l}\cdot Q_{2l-1,2l}\cdot A_{2n+1}\cdot D_{2l+1}A_{2l+1}\cdot\ldots\cdot D_{2n}A_{2n}). \tag{41}$$

We may suppose, that $A_{2n+1}=1$ in (41) (all other terms are annihilate with each other), and if

$$A_{2s-1}=\begin{cases}x_j\\ \partial_j\end{cases}$$

than

$$A_{2s}=\begin{cases}\partial_j\\ x_j\end{cases}$$

It is easy to see, that $(41) = (-1)^n \cdot (l!)^2 \cdot (2n-2l)! \cdot 2^l$. In particular, all the summands in $\Psi_{2n+1}(\partial_1, \dots, \partial_n, x_1, \dots, x_n, 1)$ contributes a nonzero number, and all these numbers have the same sign. It proves that the leading coefficient on $\boldsymbol{\lambda}$ in (38) is not equal to 0.

The general case of the cocycle Ψ_{2m+1} $(m \geq n)$ is deduced to the previous computation.

It follows from these computations and Theorem 4.2 that cocycles $\Psi_{2n+1}, \Psi_{2n+3}, \Psi_{2n+5}, \dots$ are noncohomologous to 0 cocycles in $C^\bullet_{\mathrm{Lie}}(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_n); \mathbb{C})$. It was proved in [FT1] that $H^\bullet(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_n); \mathbb{C})$ is the exterior algebra with the unique generator in each dimension $2n+1, 2n+3, 2n+5, \dots$. Using the above arguments concerning the Hopf algebra structure on $H^\bullet(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_n); \mathbb{C})$ we prove the following theorem:

Theorem.

$$H^\bullet(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_n); \mathbb{C}) \simeq \wedge^\bullet(\Psi_{2n+1}, \Psi_{2n+3}, \Psi_{2n+5}, \dots).$$

□

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IUM, 11 Bol'shoj Vlas'evskij per., Moscow 121002, Russia
E-mail address: borya@mccme.ru